Abstract

Gibbard [Gibbard, A., 1973. Manipulation of voting schemes: a general result. Econometrica 41, 587–602] and Satterthwaite [Satterthwaite, M., 1975. Strategy-proofness and Arrow’s conditions: existence and correspondence theorems for voting procedures and social welfare functions. Journal of Economic Theory 10,187–217] show that an anonymous social choice function with more than two alternatives in its range must be manipulable. Under the constraint that the number of agents is larger than the number of alternatives if the latter is four, and larger than this number plus one if it is at least five, we derive the lower bound on the number of manipulable profiles of such social choice functions. Moreover, all such social choice functions attaining this lower bound are characterized. These social choice functions exhibit a trade off between minimizing manipulability and treating alternatives neutrally.

1. Introduction

A well-known result of Gibbard (1973) and Satterthwaite (1975) states that any strategy-proof social choice function with more than two alternatives in its range is dictatorial. This implies that anonymity is not compatible with strategy-proofness. Here we investigate how incompatible
these conditions are, specifically, how many manipulable profiles of preferences an anonymous social choice function has to admit.

The question of minimal manipulability in this specific sense – by counting the number of manipulable profiles – was pioneered by Kelly (1988), who finds that this minimum for nondictatorial social choice functions is equal to two in case there are three alternatives and two agents. Among the social choice functions reaching this minimum there are attractive ones, satisfying anonymity and Pareto-optimality. Fristrup and Keiding (1998) derive the minimal number for two agents and more than three alternatives. Among the nondictatorial social choice functions reaching the lower bound are the so-called almost dictatorial ones, which are dictatorial except on exactly one profile, and certainly not anonymous. Maus et al. (in press-c) show that under the additional condition of unanimity these almost dictatorial social choice functions are exactly the minimally manipulable nondictatorial ones if both the number of agents and the number of alternatives are at least three. They also show that for three alternatives and any number of agents the lower bound for manipulable profiles under nondictatoriality alone is equal to the number of agents, and this lower bound is reached by six social choice functions. These social choice functions are also anonymous.

In the present paper we consider anonymous social choice functions for more than three alternatives. We derive the lower bound for the number of manipulable profiles and identify the social choice functions attaining this lower bound, for two cases: either the number of alternatives is four and the number of agents is at least five, or the number of alternatives is greater than four and the number of agents is at least two more than the number of alternatives. The remaining cases are still open, and it seems that they cannot be handled by the methods employed in this paper. The social choice functions attaining the lower bound are extensions of the ones mentioned at the end of the preceding paragraph. These social choice functions show how one can (but probably would not want to) minimize manipulability while treating agents anonymously. They ‘minimize surjectivity’ – loosely speaking – in the sense that except for two alternatives, say $b$ and $c$, for all other alternatives there is precisely one profile at which they are chosen. At the remaining profiles the choice between $b$ and $c$ is on a unanimity basis: $c$ is chosen unless $b$ is unanimously preferred to $c$. Thus, alternatives are treated in a non-neutral way, which is usually regarded equally undesirable as treating agents unequal. Nevertheless, the result shows what is possible if we want to minimize manipulability.

Within the mentioned restrictions on the numbers of alternatives and agents, we show as a main step in our proof of the main result that strategy-proof and anonymous social choice between more than two alternatives is not possible on subsets of the domain of preference profiles satisfying a certain diversity condition, namely on subsets where agents hold at least $k \in \mathbb{N}$ different preferences. We point this out here since this is an impossibility result on a restricted domain that is of interest on its own. It says that the impossibility result of Gibbard and Satterthwaite cannot be resolved by demanding diversity of preferences. The reader who is familiar with a specific proof of Arrow’s theorem (Arrow, 1963) and a proof Gibbard–Satterthwaite’s theorem building on Arrow’s theorem will recognize a lot of similarities. The basic idea is to show that the steps taken there can be adapted to go through also on the restricted domains considered here. The restrictions on the number of agents compared to the number of alternatives in our main result are needed to ensure that these steps can be adapted.

Other work on minimal manipulability in the sense of this paper includes Kelly (1989, 1993), Aleskerov and Kurbanov (1999), Slinko (2002), and Maus et al. (in press-a,b). For brevity we omit detailed reviews of these contributions.
After preliminaries in Section 2 we link strategy-proofness to monotonicity, Pareto-optimality and decisiveness on sets of profiles where agents hold at least \( k \) different preferences, in Sections 4 and 5. In Section 6 we combine this to show that there are no strategy-proof and anonymous social choice functions selected among more than two alternatives on these restricted domains. Section 7 contains the main result of the paper: We characterize the minimally manipulable anonymous social choice functions selecting from more than three alternatives. Section 8 concludes.

2. Preliminaries

Let \( A \) be a finite set of alternatives, \( m := |A| \geq 3 \), and let \( N \) a finite set of agents, \( n := |N| \geq 2. \)

Let \( t \subseteq A \times A \) be a binary relation on \( A \). We call \( t \) complete if \((x, y) \in t\) or \((y, x) \in t\) for all \( x, y \in A \). Note that completeness of \( t \) implies \((x, x) \in t\) for all \( x \in A \). We call \( t \) transitive if \((x, y) \in t\) and \((y, z) \in t\) implies \((x, z) \in t\) for all \( x, y, z \in A \). We call \( t \) antisymmetric if \((x, y) \in t\) and \((y, x) \in t\) implies \(x = y\), for all \( x, y \in A \).

A preference \( t \subseteq A \times A \) is a linear ordering on \( A \), i.e., a complete, transitive, and antisymmetric binary relation. Let \( P \) denote the set of all preferences. Suppose that \( A = \{x_1, x_2, \ldots, x_m\} \). By completeness, transitivity and antisymmetry we can write conveniently

\[
t = x_1x_2\ldots x_m
\]

for the preference \( t \) such that \((x_i, x_j) \in t\) if and only if \(i \geq j\), \(i, j \in \{1, 2, \ldots, m\}\), and

\[
t = \ldots x_i \ldots y \ldots
\]

if we want to express only that \( x \) is strictly preferred to \( y \).

A profile \( p \) is a map from \( N \) to \( P \). Thus, \( P^N \) denotes the set of all profiles. A profile \( p \) assigns to every agent \( i \) a preference \( p(i) \) over the alternatives. For a nonempty subset \( S \) of \( N \) we denote by \( p|_S \) the restriction of the map \( p \) to the domain \( S \). We denote a profile such that all agents have the same preference \( t \in P \) by \( t^N \).

A social choice function is a surjective function \( f: P^N \rightarrow A \). Hence, a social choice function selects a unique alternative \( f(p) \) at every profile \( p \).

For a permutation \( \sigma \) of \( N \) and a profile \( p \in P^N \) let \( p \circ \sigma \) be the profile given by \((p \circ \sigma)(i) := p(\sigma(i))\) for all \( i \in N \). A social choice function is called anonymous if \( f(p) = f(p \circ \sigma) \) for all permutations \( \sigma \) of \( N \). Thus, anonymous social choice functions are symmetric in the arguments: They treat agents equally.

In contrast to anonymity, the following dictatorial social choice functions \( \text{dict}_d \) respect only the preference of one single agent \( d \in N \), the dictator. For any profile \( p \) \( \text{dict}_d \) is defined by

\[
\text{dict}_d(p) := x
\]

where \( x \) is such that \( p(d) = x \)… So, \( \text{dict}_d(p) \) is the most preferred alternative of agent \( d \) in \( p(d) \). A social choice function \( f \) is called nondictatorial if there is no agent \( d \) such that \( f = \text{dict}_d \).

3. Manipulation of social choice functions

We are interested in strategic behavior of individuals when facing cooperative decision-making as captured by social choice functions. This is formalized by the following definitions.

\( ^2 \)We denote the cardinality of a set \( S \) by \(|S|\).
Let \( f : P^N \rightarrow A \) be a social choice function. Let \( p \in P^N \) be a profile. Then a profile \( q \) such that, for some \( i \in N \), \( q|_{N \setminus \{i\}} = p|_{N \setminus \{i\}} \) and \( q(i) \neq p(i) \), is called an \( i \)-deviation or simply a deviation from \( p \). Letting \( t := q(i) \) we also use the notation \( q = (p - i, t) \). A profile \( p \) is called manipulable (under \( f \)) if there is an agent that is better off by being dishonest about his preference, i.e., if there is an \( i \in N \) and an \( i \)-deviation \( q \) such that

\[
(f(p), f(q)) \not\in p(i).
\]

In this case we say that \( p \) is manipulable towards \( q \) (under \( f \)). Letting \( B \subseteq P^N \) and

\[
M_f(B) := \{ p \in B \mid p \text{ is manipulable towards some } q \in B \text{ under } f \},
\]

A social choice function is called strategy-proof on \( B \) if

\[
M_f(B) = \emptyset,
\]

otherwise it is said to be manipulable on \( B \). If \( B = P^N \) we omit \( B \) and write \( M_f := M_f(P^N) \).

The prominence of the dictatorial rules arises from the following impossibility result due to Gibbard (1973) and Satterthwaite (1975).

**Theorem 3.1.** Let \( f \) be a nondictatorial social choice function. Then \( |M_f| \geq 1 \).

This theorem makes no statement about the number of manipulable profiles such social choice functions admit. This question has been solved by Kelly (1988) for two agents and three alternatives, and by Fristrup and Keiding (1998) for two agents, and any number of alternatives larger than three. In Maus et al. (in press-c) it is shown that for \( m = 3 \) and any number of agents larger than two, the following social choice functions are exactly the minimally manipulable ones among the nondictatorial social choice functions.

**Definition 3.2.** Choose \( b, c \in A \), \( b \neq c \), and let \( A = \{a_1, \ldots, a_{m-2}, b, c\} \). Choose \( m - 2 \) different preferences \( t_j \in P, j \in \{1, 2, \ldots, m-2\} \), such that \( t_j = \ldots a_j \ldots b \ldots c \ldots \) for all \( j \in \{1, 2, \ldots, m-2\} \). Let \( \mu : P^N \rightarrow A \) be the social choice function given by

\[
\mu(p) := \begin{cases} 
  a_j & \text{if } p = t_j^N \text{ for some } j \in \{1, 2, \ldots, m-2\} \\
  b & \text{if } (b, c) \in p(i) \text{ for all } i \in N \text{ and } p \not\in \{t_1^N, \ldots, t_{m-2}^N\} \\
  c & \text{if there is an } i \in N \text{ such that } (c, b) \in p(i).
\end{cases}
\]

Note that the social choice functions given by Definition 3.2 are not only nondictatorial but even anonymous. Thus, they are the minimally manipulable anonymous social choice functions for three alternatives. In this paper we show that also for \( m \geq 4 \) the social choice functions given by Definition 3.2 are the minimally manipulable anonymous social choice functions. One might hope that, just as in the case \( m = 3 \), the minimally manipulable nondictatorial social choice functions coincide with the anonymous ones. This, however, is not true, see Maus et al. (in press-c) and the end of Section 7 below.

4. Monotonicity and Pareto-optimality on \( B^k \)

Let \( k \in \mathbb{N} \) and let

\[
B^k := \{ p \in P^N \mid |p(N)| \geq k \}
\]

be the set of all profiles that contain at least \( k \) different preferences. We want to show an impossibility result for strategy-proof and anonymous social choice functions on \( B^k \) for certain \( k \).
This section contains two ingredients for that impossibility result, a monotonicity lemma and a Pareto-optimality lemma.

Let \( p, q \in P^N \) and \( a \in \bar{A} \subseteq A \). We say that a weakly improves from \( p \) to \( q \) with respect to \( \bar{A} \), if for all alternatives \( x \in \bar{A} \), we have for all agents \( i \in N \)

\[
(a, x) \in p(i) \Rightarrow (a, x) \in q(i).
\]

Let \( f: P^N \to A \) be a social choice function and let \( B \subseteq P^N \). We say that \( f \) is a monotone on \( B \) if for all \( p, q \in B \) such that \( f(p) \) weakly improves from \( p \) to \( q \) with respect to \( f(B) \) we have

\[
f(p) = f(q).
\]

This monotonicity condition corresponds to the one used in Dasgupta et al. (1979) for social choice rules, and is also known as Maskin-monotonicity. The following lemma links strategy-proofness to monotonicity.

**Lemma 4.1.** Let \( f: P^N \to A \) be anonymous and strategy-proof on \( B^k \), \( k \in \mathbb{N} \). Then \( f \) is monotone on \( B^k \).

**Proof.** Let \( p, q \in B^k \) be such that \( f(p) \) weakly improves from \( p \) to \( q \) with respect to \( f(B^k) \). We have to show that \( f(p) = f(q) \).

**Claim 1.** If there are \( i, j \in N \), \( i \neq j \), such that \( p(i) = q(j) \), then \( f(p) \) weakly improves from \( p \) to \( q \), where \( \vec{q}_i = q_j \), \( \vec{q}_j = q_i \), and \( \vec{q}_k = q_k \) for all \( k \in \mathbb{N} \backslash \{i, j\} \).

**Proof.** For all \( l \in N \backslash \{i, j\} \) we have

\[
(f(p), x) \in p(l) \text{ implies } (f(p), x) \in \vec{q}(l) \text{ for all } x \in f(B^k), \tag{1}
\]

since \( \vec{q}(l) = q(l) \) and \( f(p) \) weakly improves from \( p \) to \( q \) with respect to \( f(B^k) \). For \( l = i \) we have \( p(l) = p(i) = q(j) = \vec{q}(i) = \vec{q}(l) \), so that (1) also holds for \( l = i \). For \( l = j \) and all \( x \in f(B^k) \), \( (f(p), x) \in p(l) \) implies \( (f(p), x) \in q(l) = q(j) = p(l) \), since \( f(p) \) weakly improves from \( p \) to \( q \) with respect to \( f(B^k) \). In turn, \( (f(p), x) \in p(i) \) implies \( (f(p), x) \in q(i) = \vec{q}(j) = \vec{q}(l) \). Thus, (1) holds for all \( l \in N \), so \( f(p) \) weakly improves from \( p \) to \( q \) with respect to \( f(B^k) \). This proves Claim 1.

Note that \( p(i) = \vec{q}(i) \) for \( p \) and \( q \) as in Claim 1, and that \( \vec{q} \in B^k \). By repeated application of Claim 1 we obtain a \( \vec{q} \in B^k \) such that \( p(i) \neq \vec{q}(j) \) for all \( i, j \in S := \{k \in N \mid p(k) \neq \vec{q}(k)\} \), i.e., \( p(S) \cap \vec{q}(S) = \emptyset \). By anonymity \( f(q) = f(\vec{q}) \), so it suffices to show that \( f(\vec{q}) = f(p) \).

Furthermore, by anonymity we may assume that \( S = \{1, 2, \ldots, s\} \), \( s = |S| \), and we may number the agents in \( S \) such that for some largest possible \( v \in \{1, 2, \ldots, s\} \) we have \( \vec{q}(j) \subseteq \vec{q}((N \setminus S) \cup \{1, \ldots, j - 1\}) \) for all \( j \in \{1, \ldots, v - 1\} \) (where the latter set is defined empty for \( v = 1 \)).

Let the profiles \( r^l \in P^N \), \( l \in \{0, \ldots, s\} \), be defined by

\[
r^l(i) := \begin{cases} \vec{q}(i) & i \leq l \\ p(i) & i > l \end{cases}
\]

for all \( i \in N \).

**Claim 2.** \( r^l \in B^k \) for all \( l \in \{0, \ldots, s\} \).

**Proof.** The proof is by induction on \( j \). For \( j = 0 \) the claim follows since \( r^0 = p \in B^k \). Suppose now that \( |r^{j-1}(N)| \geq k \) for some \( j \in \{1, \ldots, s\} \). We have to show that \( |r^j(N)| \geq k \). If \( j < v \), then \( \vec{q}(j) \subseteq \vec{q}((N \setminus S) \cup \{1, \ldots, j - 1\}) \) for all \( j \in \{1, \ldots, v - 1\} \) (where the latter set is defined empty for \( v = 1 \)).

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Let the profiles \( r^l \in P^N \), \( l \in \{0, \ldots, s\} \), be defined by

\[
r^l(i) := \begin{cases} \vec{q}(i) & i \leq l \\ p(i) & i > l \end{cases}
\]

for all \( i \in N \).

**Claim 2.** \( r^l \in B^k \) for all \( l \in \{0, \ldots, s\} \).
\(q((N \setminus S) \cup \{1, \ldots, j-1\})\). Furthermore, since \(p(S) \cap q(S) = \emptyset\) and \(j \in S\) we have \(q(j) \not\in p(\{j+1, \ldots, s\})\). Since

\[
\mathcal{r}^l(N) = q(\{1, \ldots, j-1\}) \cup \{q(j)\} \cup p(\{j+1, \ldots, s\}) \cup p(N \setminus S)
\]

\[
= q(\{1, \ldots, j-1\}) \cup \{q(j)\} \cup p(\{j+1, \ldots, s\}) \cup q(N \setminus S)
\]

it follows that \(q(j) \not\in \mathcal{r}^l(N \setminus \{j\})\). Since \(\mathcal{r}^{j-1}(N \setminus \{j\}) = \mathcal{r}^l(N \setminus \{j\})\), \(q(j) \not\in \mathcal{r}^l(N \setminus \{j\})\) and \(|\mathcal{r}^{j-1}(N)| \geq k\), hence \(|\mathcal{r}^l(N)| \geq k\). To complete the proof of the induction step suppose \(j \geq v\). By the maximality of \(v\) we have \(q(i) \in q((N \setminus S) \cup \{1, \ldots, v-1\})\) for all \(i \in \{v, \ldots, s\}\). Hence

\[
\mathcal{q}(N) = \mathcal{q}(S) \cup \mathcal{q}(N \setminus S)
\]

\[
= \mathcal{q}(\{1, \ldots, v-1\}) \cup \mathcal{q}(\{v, \ldots, s\}) \cup \mathcal{q}(N \setminus S)
\]

\[
= \mathcal{q}(\{1, \ldots, v-1\}) \cup \mathcal{q}(N \setminus S)
\]

\[
= \mathcal{r}^l(\{1, \ldots, v-1\}) \cup \mathcal{r}^l(N \setminus S)
\]

\[
\subseteq \mathcal{r}^l(N),
\]

and therefore \(\mathcal{q} \in B^k\) implies \(\mathcal{r}^l \in B^k\). This completes the proof of Claim 2.

We can now show \(f(\mathcal{q}) = f(p)\). Note that \(\mathcal{r}^0 = p\), \(\mathcal{r}^n = \mathcal{q}\), and by Claim 2, \(\mathcal{r}^l \in B^k\) for all \(l \in \{0, \ldots, n\}\). Since \(f\) is strategy-proof on \(B^k\) we have \((f(\mathcal{r}^0), f(\mathcal{r}^1)) \in \mathcal{r}^l(1) = \mathcal{q}(1)\) and \((f(\mathcal{r}^0), f(\mathcal{r}^0)) \in p(1)\). Since \(f(\mathcal{r}^0) = f(p)\) and \(f(p)\) weakly improves from \(p\) to \(\mathcal{q}\) with respect to \(f(B^k)\), \((f(\mathcal{r}^0), f(\mathcal{r}^0)) \in p(1)\) implies \((f(\mathcal{r}^0), f(\mathcal{r}^0)) \in \mathcal{q}(1)\). So, \((f(\mathcal{r}^1), f(\mathcal{r}^0)) \in \mathcal{r}^l(1) = \mathcal{q}(1)\) and \((f(\mathcal{r}^0), f(\mathcal{r}^1)) \in \mathcal{q}(1)\).

This implies \(f(p) = f(\mathcal{r}^0) = f(\mathcal{r}^l)\) since preferences are antisymmetric. Repeating this argument yields \(f(p) = f(\mathcal{r}^1) = f(\mathcal{r}^2) = \ldots = f(\mathcal{r}^n) = f(\mathcal{r})\), which completes the proof. \(\square\)

Lemma 4.1 entails the following corollary.

**Corollary 4.2.** Let \(f: P^N \to A\) be anonymous and strategy-proof on \(B^k\), \(k \in \mathbb{N}\). Let \(a \in f(B^k)\) and \(p \in B^k\) be such that \((a, x) \in p(i)\) for all \(i \in \mathbb{N}\) and all \(x \in f(B^k)\). Then \(f(p) = a\).

Let \(x, y \in A\), \(x \neq y\). We say that \(x\) Pareto dominates \(y\) at the profile \(p\) if \((x, y) \in p(i)\) for all \(i \in \mathbb{N}\). A social choice function is called Pareto optimal on \(B \in P^N\), if it does not choose alternatives Pareto dominated by an element of \(f(B)\), i.e., for each \(p \in B\) there is no \(x \in f(B) \setminus \{f(p)\}\) such that \(x\) Pareto dominates \(f(p)\) at \(p\).

The second lemma shows that if \(f\) is anonymous and strategy-proof on \(B^k\), then \(f\) is Pareto optimal on \(B^k\). We have to make restrictions on \(k\) in order to make sure that there are sufficiently many different preferences having the same alternative at the first and second places.

**Lemma 4.3.** For \(m = 4\) let \(k \leq m + 1\) and for \(m \geq 5\) let \(k \leq m + 2\). Let \(f: P^N \to A\) be anonymous and strategy-proof on \(B^k\). Then \(f\) is Pareto optimal on \(B^k\).
**Proof.** Let \( p \in B^k \), \( x \in f(B^k) \) and \( y \in A \) be such that \( x \) Pareto dominates \( y \) at \( p \). Suppose that \( f(p) = y \). It is sufficient to derive a contradiction. By Lemma 4.1, \( f \) is monotone on \( B^k \). Since \( p \in B^k \) and \( f \) is anonymous we may without loss of generality assume that \( |p(\{1, \ldots, k\})| = k \). Let \( l := \max \{k - (m - 2), 0\} \), then by monotonicity we can assume that for \( l < i \leq k \) we have \( p(i) = xy \). and, moreover, that \( p(i) = xy \) for all \( i > k \). If \( l = 0 \) then \( f(p) = x \) by Corollary 4.2, a contradiction. Hence, \( l \geq 1 \). For \( i \in \{1, \ldots, l\} \) let \( Z_i := \{a \in A \mid (a, y) \in p(i)\} \) be the upper contour of \( y \) at \( p(i) \). There are precisely \( (m - 2)! \) preferences where \( x \) is best and \( y \) is second best and \( l \geq 1 \), hence \( k > (m - 2)! \) and \( \{x, y\} \subseteq Z_i \) for all \( i \in \{1, \ldots, l\} \). Consider \( r(i) \) obtained from \( p(i) \) by shifting alternative \( x \) to the top while leaving all other alternatives unchanged, formally:

\[
r(i) = (\{x\} \times A) \cup (p(i) \cap (A \setminus \{x\})^2).
\]

For \( l < i \leq k \) we have \( r(i) = p(i) \). For the agents smaller than or equal to \( l \) we consider two cases.

**Case 1.** \( |\{Z_1, \ldots, Z_l\}| = l \).

In this case, \( \{x, y\} \subseteq Z_i \) and \( |\{Z_1, \ldots, Z_l\}| = l, |r(\{1, \ldots, k\})| = k \). Hence, \( r \in B^k \), so that \( f(r) = y \) by monotonicity. On the other hand, by Corollary 4.2 we have \( f(r) = x \), a contradiction.

**Case 2.** \( |\{Z_1, \ldots, Z_l\}| < l \).

In this case, \( l \geq 2 \). But \( l = k - (m - 2)! \leq (m + 2) - (m - 2)! \leq 1 \) for \( m \geq 5 \). So, \( m = 4 \) and \( l \in \{2, 3\} \).

Without loss of generality \( Z_1 = Z_2 \). Let \( A = \{x, y, a, b\} \). Since \( \{x, y\} \subseteq Z_i \) for all \( i \in \{1, \ldots, l\} \), there are alternatives \( z_1 \in Z_1 \setminus \{x, y\} \) and, if \( l = 3 \), \( z_3 \in Z_3 \setminus \{x, y\} \). If \( l = 3 \) and \( Z_3 = Z_1 = Z_2 \) we must have \( |Z_1| = 4 \) since \( |Z_1| = 3 \) implies that without loss of generality \( p(\{1, 2, 3\}) \subseteq \{axyb, axby\} \), contradicting \( |p(\{1, \ldots, k\})| = k \). So, if \( l = 3 \), we can choose a \( z_3 \in Z_1 \setminus \{x, y, z_1\} \). Without loss of generality suppose that \( z_1 = a \) and, if \( l = 3 \), \( z_3 = b \). Consider the profile \( \bar{r} \) defined by

\[
\bar{r}(i) := \begin{cases} 
axyb & \text{if } i = 1 \\
axby & \text{if } i = 2 \\
xbya & \text{if } i = l = 3 \\
r(i) & \text{if } i > l.
\end{cases}
\]

Then \( \bar{r} \in B^k \) and \( y \) weakly improves from \( p \) to \( \bar{r} \). Hence, \( f(\bar{r}) = y \). Let \( \hat{r} := (r_{-1}, xaby) \). By Corollary 4.2, \( f(\hat{r}) = x \). So, \( \hat{r} \) is a 1-deviation from \( \bar{r} \) such that \( (f(\bar{r}), f(\hat{r})) \notin \bar{r}(1) \). But then \( \bar{r} \in B^k \) is manipulable. This contradiction completes the proof. \( \square \)

5. **Decisiveness on** \( B^k_{\{a, b\}} \)

Let \( 0 < k \leq n \), let \( X \subseteq A \), and let \( B^k_X \) be the set of all profiles in \( B^k \) where all alternatives in \( X \) are strictly preferred to alternatives in \( A \setminus X \), i.e.,

\[
B^k_X := \{ p \in B^k \mid \forall i \in N \ \forall x \in X \ \forall y \in A \setminus X \ [(x, y) \in p(i)] \}.
\]

Let \( p \in B^k_{\{a, b\}}, a \neq b \). We say that \( S \subseteq N \) decides over \( (a, b) \) at \( p \) if \( f(p) = a \) and \( S = \{i \in N \mid (a, b) \in p(i)\} \). We say that \( S \) decides over \( (a, b) \) if \( S \) decides over \( (a, b) \) at all profiles \( p \in B^k_{\{a, b\}} \) that satisfy \( S = \{i \in N \mid (a, b) \in p(i)\} \). We denote the set of all \( S \) that are decisive over \( (a, b) \) by \( W^k(a, b) \). The following is an immediate consequence of the definitions of decisiveness, weak improvement and monotonicity.
Remark 5.1. For social choice functions $f$ which are monotone on $B^k$, $S$ decides over $(a, b)$ at a profile $p \in B^k_{(a,b)}$ if and only if $S \subseteq W^k (a, b)$.

The next lemma shows some decisiveness properties of $f$ on $B^k$ over triples of alternatives in case $f$ is strategy-proof on $B^k$.

Lemma 5.2. Let $k \leq m$ if $m = 4$ and $k \leq m + 2$ if $m \geq 5$. Let $f$ be anonymous and strategy-proof on $B^k$, and assume $f (B^k) \supseteq \{a, b, c\}$ for different $a, b$ and $c$. Let $S \subseteq W^k (a, b)$. Then

(i) $S \subseteq W^k (c, b)$ and $S \subseteq W^k (a, c)$
(ii) $S \subseteq W^k (b, a)$
(iii) $|S| > \frac{1}{2} n$.

Proof. (i) By Lemma 4.1 $f$ is monotone on $B^k$. Then $B^k_{(a,b)}$ is nonempty by the restrictions on $k$. Let $p \in B^k_{(a,b)}$ be such that $S$ is decisive over $(a, b)$ at $p$. For a permutation $\tau$ on $A$ and a preference $t \in P$ let

$$t^\tau = \{ ((\tau(x), \tau(y))) | (x,y) \in t \}.$$

For $x, y \in A$ and $t \in P$ let $t^{(xy)} \in P$ denote the preference that switches $x$ and $y$ while leaving all other alternatives invariant.

Let $p^1$ be such that $p^1 |S|=p|S|$ and $p^1 (i) = p(i) (ac)$ for all $i \in N \setminus S$. Then $p^1 \in B^k$. By monotonicity $f (p^1) \neq b$, since $b$ weakly improves from $p^1$ to $p$ and $f (p) = a \neq b$. As $b$ Pareto dominates all $x \in A \setminus \{a, b\}$ at $p^1$, Lemma 4.3 implies $f (p^1) \in A \setminus \{a, b\}$, so $f (p^1) = a$.

Let $p^2$ be such that $p^2 |N \setminus S| = p^1 |N \setminus S|$ and $p^2 (i) = p^1 (i) (bc)$ for all $i \in S$. Then $p^2 \in B^k$. By monotonicity $f (p^2) = a$, since $a$ weakly improves from $p^1$ to $p^2$ and $f (p^1) = a$.

Let $p^3$ be such that $p^3 |N \setminus S| = p^2 |N \setminus S|$ and $p^3 (i) = p^2 (i) (ac)$ for all $i \in S$. Then $p^3 \in B^k$. Since $c \in f (B^k)$ Pareto dominates all alternatives $x \in A \setminus \{b, c\}$, Lemma 4.3 implies $f (p^3) \in \{b, c\}$. By monotonicity $f (p^3) \neq b$, since $b$ weakly improves from $p^3$ to $p^2$ and $f (p^2) = a \neq b$. So, $f (p^3) = c$.

Finally, let $p^4$ be such that $p^4 |N \setminus S| = p^3 |N \setminus S|$ and $p^4 (i) = p^3 (i) (ab)$ for all $i \in S$. Then $p^4 \in B^k_{(a,c)}$ and by monotonicity $f (p^4) = c$. So, $S$ decides over $(c, b)$ at $p^4$ and by Remark 5.1

$$S \subseteq W^k (c, b).$$

In order to prove that $S$ decides over $(a, c)$ let $p^5$ be such that $p^5 |S| = p^4 |S|$ and $p^5 (i) = p^4 (i) (bc)$ for all $i \in N \setminus S$. Then $p^5 \in B^k$ and since $f (p^5) = a$ we have by monotonicity that $f (p^5) = a$.

Let $p^6$ be such that $p^6 |S| = p^5 |S|$ and $p^6 (i) = p^5 (i) (ab)$ for all $i \in N \setminus S$. Then $p^6 \in B^k_{(a,c)}$ and by monotonicity $f (p^6) = a$. So, $S$ decides over $(a, c)$ at $p^6$ and by Remark 5.1

$$S \subseteq W^k (a, c).$$

(ii) By (i) we have $S \subseteq W^k (a, c)$. Thus, applying (i) to $S \subseteq W^k (a, c)$ yields $S \subseteq W^k (b, c)$ and then applying the second part of (i) to $S \subseteq W^k (b, c)$ yields $S \subseteq W^k (b, a)$.

(iii) To the contrary suppose that $|S| \leq \frac{1}{2} n$. By anonymity we have $T \subseteq W^k (a, b)$ for all $T$ such that $|T| = |S|$. By (ii) also $T \subseteq W^k (b, a)$ for all such $T$. By monotonicity this implies $T \subseteq W^k (b, a)$ for all $T$ with $|T| \geq |S|$. Thus, $N \setminus S \subseteq W^k (b, a)$ and this contradicts $S \subseteq W^k (a, b)$. \[\square\]

In case $m = 4$ and $k = m + 1 = 5$, the set $B^k_{(a,b)}$ is empty but an almost similar result as in Lemma 5.2 can be achieved using the following terminology. Let $A = \{a, b, c, d\}$. Let $B^k_{(a,b)}$ be the set of all profiles $p$ in $B^k$ where except one agent $i \in N$ prefer all alternatives in $\{a, b\}$ to all alternatives in $\{c, d\}$, and $p(i) \in \{abcd, bdac\}$. Let $p \in B^k_{(a,b)}$, $a \neq b$. We say that $S \subseteq N$ $d$-decides over $(a, b)$ at $p$ if $f (p) = a$ and $S = \{i \in N | (a, b) \in p(i) \setminus \{ip\}\}$. We say that $S$ $d$-decides over $(a, b)$ if $S$
d-decides over \((a, b)\) at all profiles \(p \in B_{[a,b]}^{kd}\) that satisfy \(S = \{ i \in N \mid (a, b) \in p(i) \} \setminus \{ i_p \}\). We denote the set of all \(S\) that are \(d\)-decisive over \((a, b)\) by \(W^d_{(a,b)}(a, b)\).

As before we make the following observation.

**Remark 5.3.** For social choice functions \(f\) that are monotone on \(B^k\), \(S\) \(d\)-decides over \((a, b)\) at a profile \(p \in B_{[a,b]}^{kd}\) if and only if \(S \in W^d_{(a,b)}(a, b)\).

**Lemma 5.4.** Let \(A = \{ a, b, c, d \}\) and \(k = 5\). Let \(f\) be anonymous and strategy-proof on \(B^k\), \(f(B^k) \supseteq \{ a, b, c \}\) for different \(a, b\) and \(c\). Let \(S \in W^d_{(a,b)}(a, b)\). Then

\[
\begin{align*}
(i) & \quad S \in W^k_{(c, b)}(c, b) \text{ and } S \in W^k_{(a, c)}(a, c), \\
(ii) & \quad S \in W^k_{(b, a)}(b, a) \text{ and } S \in W^k_{(c, a)}(c, a), \\
(iii) & \quad |S| \geq \frac{1}{2} n.
\end{align*}
\]

**Proof.** By Lemma 4.1, \(f\) is monotone on \(B^k\). Let \(S \in W^k_{(a,b)}(a, b)\). Since, by definition, \(B_{(a,b)}^{kd} \subseteq B^k\), and \(k=5\) and \(m=4\), we must have \(|p(S)| = 2\). Thus, \(2 \leq |S| \leq n - (k - |p(S)|) = n - 3\). Without loss of generality \(S = \{1, \ldots, |S|\}\). Consider the following profiles.

<table>
<thead>
<tr>
<th>(p)</th>
<th>(p^1)</th>
<th>(p^2)</th>
<th>(p^3)</th>
<th>(p^4)</th>
<th>(p^5)</th>
<th>(p^6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>{1}</td>
<td>abcd</td>
<td>abcd</td>
<td>acbd</td>
<td>cabd</td>
<td>cbad</td>
<td>acbd</td>
</tr>
<tr>
<td>(S\ {1})</td>
<td>abcd</td>
<td>abcd</td>
<td>acdb</td>
<td>cabd</td>
<td>cbda</td>
<td>acdb</td>
</tr>
<tr>
<td>(N\ {n - 1, n})</td>
<td>bacd</td>
<td>bacd</td>
<td>bcad</td>
<td>bcad</td>
<td>bcad</td>
<td>bcad</td>
</tr>
<tr>
<td>(n - 1)</td>
<td>badc</td>
<td>bcd</td>
<td>bda</td>
<td>bda</td>
<td>bda</td>
<td>bda</td>
</tr>
<tr>
<td>(n)</td>
<td>bdac</td>
<td>bdba</td>
<td>bdba</td>
<td>bdba</td>
<td>bdba</td>
<td>bdba</td>
</tr>
</tbody>
</table>

As in the proof of Lemma 5.2, using these profiles it follows that \(S \in W^k_{(c, b)}(c, b)\) and \(S \in W^k_{(a, c)}(a, c)\). Then (ii) follows by applying (i) twice, again as in the proof of Lemma 5.2. Furthermore, if \(T \subseteq N\) such that \(|T| \geq S\) and \(|T| \leq n - 3\), then by anonymity and monotonicity \(T \in W^d_{(b, a)}(b, a)\). Suppose \(S \in W^d_{(b, a)}(b, a)\) and \(|S| < \frac{1}{2} n\). Then \(|N \setminus \{n\}| = |N| - |S| - 1 > \frac{1}{2} n - 1 \geq |S|\). So, \(N \setminus \{n\} \in W^d_{(b, a)}(b, a)\). Consider the profile \(p = (p_{-n}, abcd)\). By strategy-proofness \(f(p) = a\). But \(\{i \in N \mid (b, a) \in q(i)\} = N \setminus \{n\} \in W^d_{(b, a)}(b, a)\), so \(f(p) = b\), a contradiction. \(\square\)

6. Gibbard–Satterthwaite on \(B^k\)

We combine the results of the previous sections to derive an impossibility result for strategy-proof anonymous social choice functions on the restricted domain \(B^k\), i.e., when agents’ preferences satisfy a minimal diversity condition.

**Theorem 6.1.** If \(m = 4\) let \(k \leq m + 1 \leq n\) and if \(m \geq 5\) let \(k \leq m + 2 \leq n\). Let \(f\) be anonymous and strategy-proof on \(B^k\). Then \(|f(B^k)| \leq 2\).

**Proof.** To the contrary suppose that there are different \(a, b, c \in f(B^k)\). Note that \(n \geq 5\). By Lemma 4.1, \(f\) is monotone on \(B^k\). Let \(\{S, T, U\}\) be a partition of \(N\) such that for all \(X, Y \subseteq \{S, T, U\}\) we have \(0 < |X| \leq |Y| + 1\). Consider profiles \(p\) such that

\[
\begin{align*}
p(S) & \subseteq \{ t \in P \mid t = ab \ldots \} , \\
p(T) & \subseteq \{ t \in P \mid t = bc \ldots \} , \\
p(U) & \subseteq \{ t \in P \mid t = ca \ldots \} .
\end{align*}
\]
There are \((m - 2)!\) different choices for any \(p(i)\), hence we may assume that \(p \in B^k\). Without loss of generality \(f(p) = a\). For some \(\tilde{S}\) such that \(|\tilde{S}| < \frac{1}{2} n\) we will prove that \(\tilde{S} \in W^k(a, c)\) if \((m, k) \neq (4, 5)\) and \(\tilde{S} \notin W^k_a(a, c)\) if \((m, k) = (4, 5)\). Since \(|\tilde{S}| < \frac{1}{2} n\), we have a contradiction with Lemma 5.2 or Lemma 5.4, respectively, which completes the proof.

**Case 1.** \((m, k) \neq (4, 5)\)

Let \(\{S, T, U\}\) be a partition of \(N\) such that \(S \subseteq \tilde{S}, T \supseteq \tilde{T}, U \supseteq \tilde{U}\), and for all \(X, Y \in \{\tilde{S}, \tilde{T}, \tilde{U}\}\) we have \(0 < |X| \leq \min\{|Y| + 1, |\tilde{S}|\}\). Consider \(q\) such that

\[
q(\tilde{S}) \subseteq \{t \in P | t = ac\ldots\},
\]

\[
q(\tilde{T}) \subseteq \{t \in P | t = ca\ldots\},
\]

\[
q(\tilde{U}) \subseteq \{t \in P | t = ca\ldots\}.
\]

A preference \(q(i)\) in \(\tilde{S}\) or \(\tilde{T} \cup \tilde{U}\) can be obtained in \((m - 2)!\) ways. We prove that there are such \(q \in B^k_{\{a, c\}}\). So, we have to show that

\[
k \leq \min\{|\tilde{S}|, (m - 2)!\} + \min\{|\tilde{T}| + |\tilde{U}|, (m - 2)!\} =: \alpha.
\]

There are four cases. If \(\alpha = |\tilde{S}| + |\tilde{T}| + |\tilde{U}| = |N|\) then \(\alpha = n \geq k\). If \(\alpha = |\tilde{S}| + (m - 2)!\), then, since \(|\tilde{S}| \geq 2\),

\[
\alpha \geq 2 + (m - 2)! \geq m + 2 \geq k
\]

if \(m \geq 5\), and

\[
\alpha \geq 2 + (m - 2)! = m \geq k
\]

if \(m = 4\), since \((m, k) \neq (4, 5)\). If \(\alpha = 2(m - 2)!\), then \(\alpha \geq 2 + (m - 2)!\) and we can argue as in the previous case. Since \(n \geq m + 1\) and \(m \geq 4, n \geq 5\). Thus, \(|\tilde{S}| < |\tilde{T}| + |\tilde{U}|\). Therefore the case \(\alpha = (m - 2)! + |\tilde{T}| + |\tilde{U}|\) cannot occur. So, we may assume that \(q \in B^k_{\{a, c\}}\).

Note that \(a\) weakly improves from \(p\) to \(q\), so by monotonicity \(f(q) = a\), and therefore \(\tilde{S} \in W^k_a(a, c)\). By the way \(\tilde{S}\) was chosen we have \(|\tilde{S}| < \frac{1}{2} n\), contradicting (iii) of Lemma 5.2.

**Case 2.** \((m, k) = (4, 5)\)

Subcase 2A: \(|S| = 1\)

Then \(n = 5\) by the conditions on the sizes of \(S\), \(T\), and \(U\), and by anonymity we may assume that

\[
p = (ab\ldots, bcad, bcda, cabd, cadb).
\]

Let

\[
q = (acdb, acbd, cdab, cabd, cadb).
\]

Then \(a\) has improved from \(p\) to \(q\), so \(f(q) = a\) by monotonicity. So \(\{i \in N \mid (a, c) \in q(i)\} = S = \{1, 2\} \in W^k_a(a, c)\), contradicting (iii) of Lemma 5.4.

Subcase 2B: \(|S| \geq 2\)

Let \(p^1\) be such that \(p^1|_{N \setminus S} = p|_{N \setminus S}\) and \(p^1(i) = p(i)_{\{b,c\}}\) for all \(i \in S\). Then \(p^1 \in B^k\). By monotonicity \(f(p^1) = a\), since \(a\) weakly improves from \(p\) to \(p^1\). Without loss of generality \(1 \in T\). Let \(p^2\) be equal to

\[\text{As before, } p(i)_{\{b,c\}}\text{ denotes the preference arising from } p(i)\text{ by switching } b \text{ and } c.\]
Lemma 7.1. Let \( k \geq 7 \). Minimal manipulability of anonymous social choice functions

result presented in the next section. Since the result is mainly a by-product on the road to the main

Proof. We first show (i):

\[ m! \leq (k+1)! \]

(ii) If \( 2 \leq k \leq m + l \leq n \) and \( m \geq 4 \), then

\[
\left( \frac{m!}{3} - k \right) \left( \frac{n!}{3} - 1 \right) > \left( \frac{m!}{3} - 1 \right) (m - 1)
\]

Proof. We first show (i):

\[
(k + l - 1)! = (k + l - 1) \cdots (k + l - 1 - 1) \cdot k \cdots 1
\]

\[ = (k + l - 1)! - (k + l - 1)! - (k + l - 2)! \cdots 1 \]

To show (ii), note that the inequality to be proven is equivalent to \( \frac{m!}{3} (x - 1) > kx - 1 \), where \( x := \frac{n!}{(n-k)! (m-2)!} \).

For \( m \geq 5 \) we have \( m! \geq 3(m+1)^2 \), so it is sufficient to prove that \( (m+1)^2 (x - 1) \geq kx \). In view of \( k \leq m + 1 \) it is sufficient that \( (m+1)(x - 1) \geq 1 \), which can easily be checked to be true.

If \( m = 4 \) then we have to show that \( 8(x - 1) > kx - 1 \), hence that \( 8 - kx > 7 \). If \( k = 2 \) this is equivalent to \( 6 \left( \frac{n-1}{2} \right)^2 > 7 \), which is true since \( n \geq 5 \). If \( 2 < k \leq 5 \) then also \( 8 - kx > 3 \left( \frac{n-1}{2} \right)^2 > 7 \), since \( n \geq 5 \). \( \square \)

Now we can prove the main result. It characterizes the minimally manipulable anonymous social choice functions for more than three alternatives.

Theorem 7.2. Let \( m \geq 4 \) and let \( f: P^N \to A \) be an anonymous social choice function. Let \( n \geq m + 1 \) if \( m = 4 \) and \( n \geq m + 2 \) if \( m \geq 5 \). Then

\[ |M_f| \geq \left( \frac{m!}{3} - 1 \right) (m - 2). \]

Furthermore, equality holds if and only if \( f \) is a social choice function as in Definition 3.2.
Proof. Let \( P = \{ t_1, \ldots, t_m \} \) and \( 0! = 1 \). Let \( p \in P^N \) and let \( c(p, i) := |p^{-1}(t_i)|, i \in \{1, \ldots, m! \} \). Then there are

\[
\frac{n!}{\prod_{i=1}^{m} c(p, i)!}
\]

profiles \( q \in P^N \) that satisfy \( c(q, i) = c(p, i) \) for all \( i \in \{1, \ldots, m! \} \). By anonymity, \( f(p) = f(q) \), \( p \in M_f \iff q \in M_f \), and \( p \in B_k \iff q \in B_k \) for all such profiles \( q \). Let

\[
l' := \begin{cases} m + 1 & \text{if } m = 4 \\ m + 2 & \text{if } m \geq 5. \end{cases}
\]

Suppose that

\[
|M_f| \leq n \left( \frac{m!}{3} - 1 \right) (m - 2).
\]

**Step 1:** \( B^l \cap M_f = \emptyset \).
Suppose to the contrary that there is a \( p \in B^l \cap M_f \). Since \( p \in B^l \), at least \( l \) of the \( c(p, i), i \in \{1, \ldots, m! \} \), satisfy \( c(p, i) \geq 1 \). Hence, by Lemma 7.1

\[
\frac{m!}{\prod_{i=1}^{m} c(p, i)!} \leq \left( \sum_{i=1}^{m} c(p, i) - l \right)! = (n - l)!.
\]

So,

\[
|M_f| \geq \frac{n!}{\prod_{i=1}^{m} c(p, i)!} \geq \frac{n!}{(n - l)!} = n \alpha(l),
\]

where \( \alpha(l) := (n - 1)! / (n - l)! \). If \( m = 4 \), then since \( n \geq m + 1 \),

\[
\alpha(l) = \prod_{j=1}^{l-1} (n - j) \geq \prod_{j=1}^{m} (m + 1 - j) = m!
\]

\[
= 3 \left( \frac{m!}{3} \right) \geq 2 \left( \frac{m!}{3} - 1 \right) = (m - 2) \left( \frac{m!}{3} - 1 \right),
\]

and, if \( m \geq 5 \), then, since \( n \geq m + 2 \),

\[
\alpha(l) = \prod_{j=1}^{l-1} (n - j) \geq \prod_{j=1}^{m+1} (m + 2 - j) = (m + 1)!
\]

\[
> (m + 1) \left( \frac{m!}{3} - 1 \right) > (m - 2) \left( \frac{m!}{3} - 1 \right).
\]

This contradicts

\[
|M_f| \leq n(m - 2) \left( \frac{m!}{3} - 1 \right),
\]

which completes the proof of Step 1.
**Step 2:** There are \( b, c \in A \), \( b \neq c \), such that \( f(B^2) \subseteq \{ b, c \} \).

By Step 1, \( f \) is strategy-proof on \( B' \). So, by Theorem 6.1, \( |f(B^l)| \leq 2 \). Let \( b, c \in A \), \( b \neq c \), be such that \( f(B^l) \subseteq \{ b, c \} \). Let \( k \) be the smallest number such that

\[
f(B^k) \subseteq \{ b, c \}.
\]

Then \( k \geq 2 \) by surjectivity of \( f \), and \( k \leq l \) since \( f(B^l) \subseteq \{ b, c \} \). We show that \( k = 2 \). Since \( f(B^{k-1}) \not\subseteq \{ b, c \} \), there is a \( p \in B^{k-1} \setminus B^k \) such that \( f(p) = a \in A \setminus \{ b, c \} \). Furthermore, since \( k \leq n \), and \( p \in B^{k-1} \setminus B^k \), there must be agents \( i, j \in N, i \neq j \), such that \( p(i) = p(j) \), which implies that \( p(N) = p(N \setminus \{ i \}) \). Let \( t \) be such that

\[
\{(a, c), (b, c)\} \subseteq t \text{ and } t \not\in p(N).
\]

As \( |p(N)| = k - 1 \), there are at least \( m! / 3 - (k - 1) \) such preferences \( t \). Then \( t \not\in p(N) \) implies that \( q = (p_{-i}, t) \in B^k \). So \( f(q) \in \{ b, c \}, f(p) = a \) and \( f(q), f(p) \not\in t \). Hence, any such \( q \) is manipulable. So, by anonymity

\[
|M_f| \geq \left( \frac{m!}{3} - (k - 1) \right) \frac{n!}{(n - (k - 1))!}.
\]

By Lemma 7.1 the term on the right hand side is greater than \( n(m - 2)(m! - 1) \) if \( k - 1 \geq 2 \), contradicting \( |M_f| \leq n(m - 2)(m! - 1) \). Therefore, \( k = 2 \). This proves Step 2.

Let \( a \in A \) be an alternative different from \( b \) and \( c \) as in Step 2. We define

\[
P^{a; \{b,c\}} := \{ t \in P \mid \{(a, b), (a, c)\} \subseteq t \}.
\]

Let \( A = \{ a_1, a_2, \ldots, a_{m-2}, b, c \} \). Since \( f(B^2) \subseteq \{ b, c \} \), we have

\[
f(P^N \setminus B^2) \supseteq \{ a_1, a_2, \ldots, a_{m-2} \}
\]

by surjectivity. Hence, for any \( a_j, j \in \{ 1, 2, \ldots, m - 2 \} \) there is a \( t_j \in P \) such that \( f(t_j^N) = a_j \). Let \( p' := t_j^N, j \in \{ 1, 2, \ldots, m - 2 \} \). Then for any \( t \in P^{a_j; \{b,c\}} \setminus \{ t_j \} \) and \( k \in N \) we have \( f(p'_{-k}, t) \in \{ b, c \} \) and \( f(p') = a_j \), so that \( q = (p'_{-k}, t) \) is manipulable for all \( k \in N \). There are

\[
n \cdot |P^{a_j; \{b,c\}} \setminus \{ t_j \}|
\]

such \( q \), and since \( n \geq 3 \) these manipulable profiles \( q \) are different for all \( j \in \{ 1, 2, \ldots, m - 2 \} \). So,

\[
n(m - 2) \left( \frac{m!}{3} - 1 \right) \geq |M_f| \geq n \sum_{i=1}^{m-2} |P^{a_j; \{b,c\}} \setminus \{ t_j \}|.
\]

Now, \( |P^{a_j; \{b,c\}} \setminus \{ t_j \}| \geq \frac{m!}{3} - 1 \), and equality holds if and only if \( t_j \in P^{a_j; \{b,c\}} \). So, by (2), \( t_j \in P^{a_j; \{b,c\}} \) for all \( j \in \{ 1, 2, \ldots, m - 2 \} \),

\[
|M_f| = n(m - 2) \left( \frac{m!}{3} - 1 \right),
\]

and

\[
M_f = \{ (p_{-k}^j, t) \mid k \in N, j \in \{ 1, 2, \ldots, m - 2 \}, t \in P^{a_j; \{b,c\}} \setminus \{ t_j \} \}.
\]
Step 3: Let \( b, c \) be as in Step 2. Then

\[
f(P^N \setminus \{p_1, p_2, \ldots, p^m\}) = \{b, c\}.
\]

Let \( p \in P^N \setminus \{p_1, p_2, \ldots, p^m\} \). Suppose that \( f(p) \in \{a_1, a_2, \ldots, a_m\} \). Then \( p \in P^N \setminus B^2 \) and for all \( t \in P^f\{p, \{b, c\}\} \) we have \( f(p - t) \in f(B^2) \subseteq \{b, c\} \). So \( (p - t) \in M_f \), implying that \( p \in \{p_1, p_2, \ldots, p^m\} \), since \( n \geq 3 \). With surjectivity this proves Step 3.

Step 4: Let \( p \notin \{p_1, p_2, \ldots, p^m\} \) and suppose that \( (b, c) \in t_j \) for some \( j \in \{1, 2, \ldots, m-2\} \). Then,

\[
f(p) = b \quad \text{if and only if} \quad (b, c) \in p(i) \quad \text{for all} \quad i \in N.
\]

Let \( t, \bar{t} \in P \) be such that \( t = \ldots b, \ldots, c, \ldots b, \ldots \) Let \( k \in N \). Since \( t, \bar{t} \in P^{a_0, \{b, c\}} \), \( f \) is not manipulable at \( (p^j_k, t) \in B^2 \) and \( (p^j_k, \bar{t}) \in B^2 \). This implies that \( f(p^j_k, t) = b \) and \( f(p^j_k, \bar{t}) = c \).

Suppose that there is an \( i \in N \) such that \( (c, b) \in p(i) \), without loss of generality \( i = n \). Let

\[
r^u(l) := \begin{cases} p(l) & \text{if } l \leq u \\ (p^j_n, \bar{t})(l) & \text{if } l > u, \end{cases}
\]

\( u \in \{0, \ldots, n\} \). Then \( r^0 = (p^j_n, \bar{t}), r^n = p \) and since \( \bar{t} \in P^{a_0, \{b, c\}}, r^u \in M_f \cup \{p_1, p_2, \ldots, p^m\}, u \in \{0, \ldots, n-1\} \).

Suppose that \( f(r^u) = c, u \in \{0, \ldots, n-2\} \), then by \( r^u \notin M_f, (f(r^u), f(r^{u+1})) \in r^u(u+1) = t_j \).

Since \( (b, c) \in t_j, f(r^{u+1}) \in \{b, c\} \) and \( f(r^u) = c \) it follows that \( f(r^{u+1}) = c \). Since \( f(r^0) = f(p^j_n, \bar{t}) = c \) we obtain \( f(r^{u-1}) = f(p) = c \) by induction. By nonmanipulability at \( r = r^n = (p^j_n, p(n)), (c, b) \in p(n), \) and \( f(p) \in \{b, c\} \) it follows that \( f(r^n) = c \). This shows that

\[
f(p) = b \quad \text{only if} \quad (b, c) \in p(i) \quad \text{for all} \quad i \in N.
\]

Suppose that \( (b, c) \in p(i) \) for all \( i \in N \), so in particular \( (b, c) \in p(1) \). Let

\[
r^u(l) := \begin{cases} (p^j_{l-1}, t)(l) & \text{if } l \leq u \\ p(l) & \text{if } l > u, \end{cases}
\]

\( u \in \{0, \ldots, n\} \). Then \( r^0 = p, r^n = (p^j_{l-1}, t), \) and since \( t \in P^{a_0, \{b, c\}}, r^u \in M_f \cup \{p_1, p_2, \ldots, p^m\}, u \in \{1, \ldots, n\} \).

Since \( r^u = (p^j_{l-1}, t), f(r^u) = b \). Let \( u \in \{1, 2, \ldots, n\} \) and suppose that \( f(r^u) = b \). Then by \( r^u \notin M_f, (f(r^u), f(r^{u+1})) \in r^u(u) = (p^j_{l-1}, t)(u) \in \{t, j\} \).

Since \( (b, c) \in t, (b, c) \in t_j, f(r^{u+1}) \in \{b, c\} \) and \( f(r^u) = b \) it follows that \( f(r^{u+1}) = b \). So, \( f(r^u) = f(p^j_{l-1}, t) = b \) implies by induction that \( f(r^0) = f(p) = b \). This shows that

\[
f(p) = b \quad \text{if} \quad (b, c) \in p(i) \quad \text{for all} \quad i \in N,
\]

and we have proved Step 4.

Without loss of generality, \( (b, c) \in t^1 \). Then, by Step 4, \( (b, c) \in t_j \) for all \( j \in \{1, 2, \ldots, m-2\} \).

This completes the proof. □
So-called ‘almost dictatorial’ social choice functions have \( (n - 1)\left(\frac{m!}{2} - 1\right) + 1 \) manipulable profiles, see Maus et al. (in press-c) for the definition of these social choice functions and a proof of this statement. If \( m \geq 4 \), then

\[
\begin{align*}
    n(m - 2)\left(\frac{m!}{3} - 1\right) &> n\left(\frac{m!}{2} - 1\right) \\
    &= (n - 1)\left(\frac{m!}{2} - 1\right) + \left(\frac{m!}{2} - 1\right) \\
    &> (n - 1)\left(\frac{m!}{2} - 1\right) + 1,
\end{align*}
\]

so that the minimally manipulable anonymous social choice functions have strictly more manipulable profiles than minimally manipulable nondictatorial social choice functions. This is in contrast to the three alternative cases because then

\[
\begin{align*}
    n(m - 2)\left(\frac{m!}{3} - 1\right) &= n < 2n - 1 = (n - 1)\left(\frac{m!}{2} - 1\right) + 1,
\end{align*}
\]

so the anonymous social choice functions outperform the almost dictatorial social choice functions in terms of manipulability.

We conclude this section by presenting some percentages of manipulable profiles of social choice functions, in order to give an impression in relative terms of the manipulability that has to be admitted. The following tables contain the percentages of profiles that are manipulable for the social choice functions characterized in Theorem 7.2, the almost dictatorial social choice functions, Plurality and Borda rule. The numbers for the last two are taken from Aleskerov and Kurbanov (1999), where tie-breaking according to a fixed order of alternatives is used to decide on ties. We restrict the tables to three and four alternatives. Loosely speaking, these tables indicate that there is still a lot to discover in the space between classical social choice functions and social choice functions obtained by minimizing manipulability. To avoid wrong impressions we note that Slinko (2002) has shown that the percentages of manipulable profiles of Plurality and Borda rule go to zero at a speed of at least \( O\left(\frac{1}{\sqrt{n}}\right) \).

<table>
<thead>
<tr>
<th>((m, n))</th>
<th>Almost dictatorial</th>
<th>Theorem 7.2</th>
<th>Plurality</th>
<th>Borda</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3, 2)</td>
<td>8.333333%</td>
<td>5.555556%</td>
<td>11.11%</td>
<td>38.89%</td>
</tr>
<tr>
<td>(3, 3)</td>
<td>2.314815%</td>
<td>1.38889%</td>
<td>16.67%</td>
<td>23.61%</td>
</tr>
<tr>
<td>(3, 4)</td>
<td>0.540123%</td>
<td>0.308642%</td>
<td>18.52%</td>
<td>31.02%</td>
</tr>
<tr>
<td>(3, 5)</td>
<td>0.115741%</td>
<td>0.064300%</td>
<td>23.15%</td>
<td>28.55%</td>
</tr>
<tr>
<td>(3, 6)</td>
<td>0.023577%</td>
<td>0.012860%</td>
<td>23.93%</td>
<td>27.82%</td>
</tr>
<tr>
<td>(3, 7)</td>
<td>0.004644%</td>
<td>0.002501%</td>
<td>25.73%</td>
<td>25.99%</td>
</tr>
<tr>
<td>(3, 8)</td>
<td>0.000893%</td>
<td>0.000476%</td>
<td>27.39%</td>
<td>25.99%</td>
</tr>
<tr>
<td>(3, 9)</td>
<td>0.000169%</td>
<td>0.000089%</td>
<td>27.44%</td>
<td>24.98%</td>
</tr>
<tr>
<td>(3, 10)</td>
<td>0.000031%</td>
<td>0.000017%</td>
<td>28.55%</td>
<td>24.06%</td>
</tr>
<tr>
<td>(4, 2)</td>
<td>0.1736111111111%</td>
<td>4.8611111111111%</td>
<td>20.83%</td>
<td>60.42%</td>
</tr>
<tr>
<td>(4, 3)</td>
<td>0.014467592593%</td>
<td>0.303819444444%</td>
<td>29.43%</td>
<td>51.22%</td>
</tr>
<tr>
<td>(4, 4)</td>
<td>0.000904224537%</td>
<td>0.016878858025%</td>
<td>32.47%</td>
<td>50.02%</td>
</tr>
<tr>
<td>(4, 5)</td>
<td>0.000050234697%</td>
<td>0.000879107189%</td>
<td>37.38%</td>
<td>50.44%</td>
</tr>
<tr>
<td>(4, 6)</td>
<td>0.000002616390%</td>
<td>0.000043955359%</td>
<td>38.91%</td>
<td>47.90%</td>
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<tr>
<td>(4, 7)</td>
<td>0.000000130320%</td>
<td>0.000002136719%</td>
<td>40.55%</td>
<td>46.43%</td>
</tr>
<tr>
<td>(4, 8)</td>
<td>0.000000006359%</td>
<td>0.000000101749%</td>
<td>41.33%</td>
<td>44.85%</td>
</tr>
<tr>
<td>(4, 9)</td>
<td>0.000000000303%</td>
<td>0.000000004769%</td>
<td>41.99%</td>
<td>43.15%</td>
</tr>
<tr>
<td>(4, 10)</td>
<td>0.000000000014%</td>
<td>0.00000000221%</td>
<td>41.95%</td>
<td>41.78%</td>
</tr>
</tbody>
</table>
8. Conclusion

We show that a Gibbard–Satterthwaite like result holds also on sets of profiles with a certain diversity. This is used to characterize the minimally manipulable anonymous social choice functions for more than three alternatives. It turns out that these generalize strategy-proof two alternative imputation status-quo voting, and the minimally manipulable three alternative nondictatorial social choice functions. However, contrary to the three alternative cases they are not less manipulable than the almost dictatorial social choice functions. So, they do not constitute the set of minimally manipulable nondictatorial social choice functions. Moreover, the way in which they achieve minimal manipulability is at the expense of treating alternatives unequally. This suggests that a property ensuring a more equal treatment of alternatives should be added. The natural candidate is neutrality, but unfortunately anonymity and neutrality exclude each other whenever the number of alternatives can be written as a sum of non-trivial divisors of the number of agents, see Moulin (1983, p. 25). Nevertheless, this will be an issue for further research, either demanding weaker axioms than neutrality to ensure a more equal treatment of alternatives, or weakening anonymity and demanding neutrality. Also other principles of voting could be added as properties, such as respecting unanimity, Pareto-optimality, absolute plurality, plurality or Condorcet-winners.

References


