An Impossibility Result Concerning Distributive Justice in Axiomatic Bargaining

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Abstract

A two-person bargaining solution is called endowment sensitive if in a distribution problem with money it favours the opponent of a player whose initial endowment increases. Endowment sensitivity is, thus, a fairness principle of distributive justice. We show that a bargaining solution that is endowment sensitive, scale covariant, and Pareto optimal, must be dictatorial.

1 Introduction

A two-player bargaining problem, as introduced by Nash (1950), is characterized by a feasible set of utility pairs, one of which is designated to be the disagreement outcome. The players either agree on a feasible pair in this set or they disagree, in which case the disagreement outcome results. The axiomatic approach to bargaining consists of considering a map (bargaining solution) defined on a whole collection of bargaining problems, and characterizing this map by axioms. Such axioms are assumed to capture the implicit bargaining process in an abstract manner. For instance, the Nash bargaining solution (Nash, 1950) was characterized by four axioms. Later, the Nash bargaining solution was shown to be related the Rubinstein (1982) noncooperative “solution” to the bargaining problem; this result can be seen as supporting the assumption of the Nash bargaining axioms capturing an implicit bargaining process. See also Binmore, Rubinstein and Wolinsky (1986).

The question arises whether the approach of axiomatic bargaining may be used to capture fairness principles of distributive justice. Many axioms that have been proposed in literature can be explained equally well from considerations of fairness as from bargaining principles. A convincing answer to this question, however, can only be given by considering bargaining problems derived from problems of distributive justice, instead of just abstract bargaining problems as defined above. A basic problem of distributive justice is the allocation of a good, say money, between two individuals with different preferences and different initial endowments. A basic principle of fairness may be the egalitarian principle that the individual with the lower initial income obtains a larger share of the additional

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money. The question then is whether there exists a bargaining solution satisfying this principle. In this paper we will answer this question with a qualified "no".

The problem was also addressed in Klemisch-Ahler (1982), where it is shown that most well-known bargaining solutions, including the Nash solution, will not give advantage to a bargainer if he gets poorer. Briefly, the reason is that, in the model considered by Klemisch-Ahler, a poorer individual is more risk averse, and many bargaining solutions are to the advantage of those playing against a more risk averse opponent (cf. Kihlstrom, Roth, and Schmeidler, 1981). The present paper generalizes this analysis and considers the converse problem. Specifically, we will show that under standard assumptions of axiomatic bargaining theory (Pareto optimality and Scale covariance) it is possible to accommodate the principle that a poorer person should get (weakly) preferential treatment in dividing an amount of money only if we allow dictatorship. A positive feature of this "impossibility" result is that it provides further support for the claim that the axiomatic bargaining approach implicitly captures a bargaining process. Intuitively, in a bargaining process the richer person has a stronger bargaining position.

The next section presents the impossibility result and its proof. Section 3 concludes with further discussion of related literature.

2 The impossibility result

In order to introduce the notion of a bargaining problem it is convenient to first define a bargaining pair as a pair \((f, d)\) where \(d \in \mathbb{R}^2\) and \(f : \mathbb{R} \to \mathbb{R}\) is a strictly decreasing and (not necessarily strictly) concave function with \(f(d_1) \geq d_4\). Define

\[ S(f, d) := \{ x \in \mathbb{R}^2 | x \geq d, f(x_1) \geq x_2 \}. \]

A pair \((S, d)\) is a bargaining problem if there exists a bargaining pair \((f, d)\) with \(S = S(f, d)\). Such a bargaining pair is called a bargaining pair corresponding to \((S, d)\); obviously, it is not unique. The set \(S\) is the set of feasible outcomes, and \(d\) is the disagreement outcome. The part of the graph of \(f\) which belongs to \(S\) is the Pareto optimal boundary of \(S\) and is also denoted by \(P(S)\). The interpretation of a bargaining problem is that two players either can agree on a feasible outcome in \(S\), or disagree in which case the result is \(d\). By \(B\) we denote the collection of all bargaining problems.

In order to introduce the notion of a distribution problem, let \(U\) be the set of functions \(u : \mathbb{R} \to \mathbb{R}\) which are strictly increasing, and (not necessarily strictly) concave. Note that these functions are also continuous. Members of \(U\) are interpreted as von Neumann-Morgenstern utility functions for money. Concavity is interpreted as nonincreasing marginal utility for money. A distribution problem is a 5-tuple

\[ \Gamma = (u_1, u_2, w_1, w_2, m) \]

where \(u_1, u_2 \in U\) and \(w_1, w_2, m \in \mathbb{R}_+\), the set of nonnegative real numbers. The amounts \(w_1\) and \(w_2\) denote the (initial) wealth of players 1 and 2, respectively, and \(m\) is the additional sum of money that is to be distributed. By \(G\) we denote the collection of all distribution problems.
There is a close relationship between bargaining problems and distribution problems. Firstly, let \( \Gamma = (u_1, u_2, u_1, u_2, m) \) be a distribution problem. Define

\[
S^\Gamma = \{ (u_1(w_1 + m_1), u_2(w_2 + m_2)) | m_1, m_2 \in \mathbb{R}_+, \text{ } m_1 + m_2 \leq m \}
\]

and

\[
d^\Gamma = (u_1(w_1), u_2(w_2)).
\]

Then the north-east boundary of the set \( S \), consisting of the pairs of utilities generated by distributing the whole amount \( m \) can easily be extended to the graph of a function \( f \) as in the definition of a bargaining problem, so that \( (S^\Gamma, d^\Gamma) \in B \).

Conversely, let \( (S, d) \) be a bargaining problem with a corresponding bargaining pair \( (f, d) \). Define the function \( u_1 \) on \( \mathbb{R} \) by \( u_1(t) := t \) for every \( t \in \mathbb{R} \). Define \( u_2 \) by \( u_2 := d_1 \). Define the function \( u_2 \) on \( \mathbb{R} \) by \( u_2 := d_2 \). Then \( \Gamma = (u_1, u_2, u_1, u_2, m) \) is a distribution problem and \( (S, d) = (S^\Gamma, d^\Gamma) \). For later reference, we call the distribution problem thus constructed the 1-linear distribution problem corresponding to \( (f, d) \). We conclude that every distribution problem gives rise to a bargaining problem and every bargaining problem corresponds to a distribution problem.

A bargaining solution is a map \( \phi : B \rightarrow \mathbb{R}^2 \) assigning to each bargaining problem \( (S, d) \) a feasible outcome \( \phi(S, d) \). Observe that such a bargaining solution can be used to "solve" a distribution problem by applying it to the corresponding bargaining problem and then determining a distribution of \( m \) leading to the pair of utilities assigned by \( \phi \). Implicit in this approach is that different distribution problems that have the same corresponding bargaining problem obtain the same solution as far as utilities are concerned. This approach is sometimes called the "welfarist" approach, briefly: All information needed to determine a solution should be captured by the image of the problem in utility space, viz., the bargaining problem.

Roemer (1988) calls this implicit assumption the bargaining theory axiom. He argues that if, for instance, the Nash bargaining axioms are rephrased in terms of underlying physical problems, then they are in general to weak to determine a unique bargaining solution. In other words, the implicitly assumed bargaining theory axiom is far from being innocent. He then sets forth with additional requirements on the solution in terms of the physical problem, such that the bargaining theory axiom is restored. In contrast, we adopt the welfarist approach but show that it is inconsistent with a certain ethical principle.

We call a bargaining solution \( \phi \) Pareto optimal if \( \phi(S, d) \in P(S) \) for every \( (S, d) \in B \). We call \( \phi \) scale covariant if \( \phi(aS + b, ad + b) = a\phi(S, d) + b \) for every \( (S, d) \in B \) and \( a, b \in \mathbb{R}^2 \) with \( a > 0 \), where \( aS + b := \{ ax + bz | x, z \in S \} \) and \( ax + b := (a_1x_1 + b_1, a_2x_2 + b_2) \) for all \( x, z \in \mathbb{R}^2 \). Observe that \( B \) is closed under such scale transformations. Pareto optimality is a requirement of collective rationality which seems natural, in particular if we wish to apply the solution to distribution problems as we do here. Scale covariance is a condition reflecting the assumption that the players have von Neumann-Morgenstern utility functions with non-interpersonally comparable utilities.

Our main interest is in the following property. Suppose we have two distribution problems

\[
\Gamma = (u_1, u_2, u_1, u_2, m)
\]

(1)
and

\[ \Gamma' = (u_1, u_2, w_1', w_2, m) \]  \hspace{1cm} (2)

which differ only in the fact that either \( w'_1 > w_1 \) and \( w'_2 = w_2 \) or \( w'_2 > w_2 \) and \( w'_1 = w_1 \), i.e., either player 1 or player 2 is a richer person in \( \Gamma' \). As a principle of fairness, we would like the distribution of \( m \) to change in favour of player 2 if player 1 has become richer or in favour of player 1 if player 2 has become richer in the distribution problem \( \Gamma' \). This fairness principle is expressed by the following axiom for a bargaining solution \( \phi : B \rightarrow \mathbb{R}^2 \).

**Endowment sensitivity** For all \( \Gamma, \Gamma' \) as in (1) and (2) we have:

(i) If \( w'_1 > w_1 \) and \( w'_2 = w_2 \), then \( \phi_2(S_{\Gamma'}, d_{\Gamma'}) \geq \phi_2(S_{\Gamma}, d_{\Gamma}) \).

(ii) If \( w'_1 = w_1 \) and \( w'_2 > w_2 \), then \( \phi_1(S_{\Gamma'}, d_{\Gamma'}) \geq \phi_1(S_{\Gamma}, d_{\Gamma}) \).

For \((S, d) \in B \) let \( \overline{z}(S, d) \) denote the feasible outcome with maximal first coordinate and let \( \overline{x}(S, d) \) denote the feasible outcome with maximal second coordinate.

We can now state and prove our main result.

**Theorem 1** Let the bargaining solution \( \phi : B \rightarrow \mathbb{R}^2 \) be Pareto optimal, scale covariant, and endowment sensitive. Then either \( \phi(S, d) = \overline{z}(S, d) \) for all \((S, d) \in B \) or \( \phi(S, d) = \overline{x}(S, d) \) for all \((S, d) \in B \).

Thus, Theorem 1 states that Pareto optimality, scale covariance, and endowment sensitivity combined imply dictatorship of either player 1 or player 2.

**Proof of Theorem 1** Let \((S, d) \in B \) with \( S \) non-triangular. Let \((f, d) \) be a bargaining pair, corresponding to \((S, d) \), such that \( f \) is linear on \((\infty, d_1) \), i.e., on this interval the graph of \( f \) is a negatively sloped straight line. Let

\[ \Gamma = (u_1, u_2, w_1, w_2, m) \]

be the 1-linear distribution problem corresponding to \((f, d) \), and consider the distribution problem

\[ \Gamma' = (u_1, u_2, w_1', w_2, m) \]

where \( w_2' := d_2 + m \). By the linearity of \( f \) on \((\infty, d_1) \) the set \( S_{\Gamma'} \) is a triangle with vertices \( d_{\Gamma'} = (d_1, f(d_1)) = \overline{x}(S, d), (d_1 + m, f(d_1)) = \overline{z}(S, d), \) and \((d_1, f(d_1) - m) \). By endowment sensitivity

\[ \phi_1(S_{\Gamma'}, d_{\Gamma'}) \geq \phi_1(S, d). \]  \hspace{1cm} (3)

By reversing the roles of players 1 and 2 and applying an analogous argument we obtain the existence of a bargaining problem \((S_{\Gamma''}, d_{\Gamma''}) \) with \( S_{\Gamma''} \) being a triangle with vertices \( d_{\Gamma''} = \overline{x}(S, d), \overline{z}(S, d) + \overline{x}(S, d), \) and \((\alpha, d_1) \), with \( \alpha > \overline{z}_1(S, d) \), for which by endowment sensitivity

\[ \phi_2(S_{\Gamma''}, d_{\Gamma''}) \geq \phi_2(S, d). \]  \hspace{1cm} (4)

Let \((T, d) \) be the bargaining problem with \( T \) the triangle with vertices \( d, \overline{z}(S, d), \) and \( \overline{x}(S, d) \). (Cf. Figure 1.) By (3) and scale covariance, \( \phi_1(T, d) \geq \phi_1(S, d) \). By (4) and scale covariance, \( \phi_2(T, d) \geq \phi_2(S, d) \). Hence, \( \phi(T, d) \geq \phi(S, d) \). By Pareto optimality.
of $\phi$ and non-triangularity of $S$ this implies that either $\phi(S, d) = \phi(T, d) = \underline{x}(S, d)$ or $\phi(S, d) = \phi(T, d) = \overline{x}(S, d)$. In the first case, by repeating the whole argument for any arbitrary bargaining problem, we find by scale covariance that $\phi(S', d') = \underline{x}(S', d')$ for every $(S', d') \in B$. The second case is analogous. □

If Pareto optimality or scale covariance is dropped in Theorem 1, then the conclusion of the theorem no longer holds. This is clear in the case of Pareto optimality, e.g., take the bargaining solution which assigns the disagreement outcome to any bargaining problem. Dropping Pareto optimality, however, is not very natural in a normative model of distributive justice. Dropping scale covariance may be of interest if comparison of utilities makes sense. Consider the bargaining solution $\check{\phi} : B \rightarrow \mathbb{R}^2$ defined by

$$
\check{\phi}(S, d) := \begin{cases} 
\underline{x}(S, d) & \text{if } \underline{x}_1(S, d) - d_1 \geq \overline{x}_1(S, d) - d_2 \\
\overline{x}(S, d) & \text{if } \underline{x}_1(S, d) - d_1 < \overline{x}_1(S, d) - d_2
\end{cases}
$$

The solution $\check{\phi}$ is Pareto optimal but not scale covariant. Furthermore it is endowment sensitive, as the following lemma shows.

**Lemma 1** $\check{\phi}$ is endowment sensitive.

**Proof** Let $(S, d) \in B$ and suppose $\underline{x}_1(S, d) - d_1 \geq \overline{x}_1(S, d) - d_2$, hence $\check{\phi}(S, d) = \underline{x}(S, d)$. Let

$$
\Gamma = (u_1, u_2, w_1, w_2, m)
$$
be a distribution problem with \((S, d) = (S^\Gamma, d^\Gamma)\). Let
\[
\Gamma' = (u_1, u_2, w'_1, w'_2, m)
\]
be a distribution problem where exactly one of the players has a different initial endowment. \( (S', d') = (S^\Gamma, d^\Gamma) \) be the corresponding bargaining problem. It suffices to consider the two cases where one of the players has become richer.

Firstly, suppose \(w'_1 > w_1\) and \(w'_2 = w_2\). Then \(\hat{\sigma}_1(S', d') \geq d'_2 = d_2 = \bar{\sigma}_2(S, d) = \hat{\sigma}_2(S, d)\), so the conclusion of endowment sensitivity holds for this case.

Secondly, suppose \(w'_1 = w_1\) and \(w'_2 > w_2\). Then \(\bar{\sigma}_1(S', d') - d'_1 = \bar{\sigma}_1(S, d) - d_1\). Further,
\[
\bar{\sigma}_2(S', d') - d'_2 = u_2(w'_2 + m) - u_2(w'_2) \\
\leq u_2(w_2 + m) - u_2(w_2) \\
= \bar{\sigma}_2(S, d) - d_2,
\]
where the inequality follows from concavity of \(u_2\). Therefore, \(\bar{\sigma}_2(S', d') - d'_2 \leq \bar{\sigma}_2(S', d') - d'_1\), so that \(\hat{\sigma}_1(S', d') = \bar{\sigma}_2(S', d')\). Hence
\[
\hat{\sigma}_1(S', d') = \bar{\sigma}_1(S', d') = \bar{\sigma}_1(S, d) = \hat{\sigma}_1(S, d).
\]
So also for this case the conclusion of endowment sensitivity holds.

The case \(\bar{\sigma}_1(S, d) - d_1 < \bar{\sigma}_2(S, d) - d_2\), i.e., \(\hat{\sigma}(S, d) = \bar{\sigma}(S, d),\) is analogous. □

3 Concluding remarks

Luce and Raiffa (1957) already remarked that the Nash bargaining solution does not favour a poor man when bargaining against a rich man, assuming that the poor man has a concave utility function while the rich man has a linear utility function. They point out (op. cit., p. 130) that the game-theoretic solution concept should reflect the strategic aspects of the situation: ethical considerations should be captured by the utility functions. The present paper may be regarded as support for the claim that axiomatic bargaining theory defined on the informational basis of the traditional Nash model satisfies the requirements put forward by Luce and Raiffa. Luce and Raiffa want the individual utility functions to reflect ethical considerations. This would mean that the arguments of these functions should include not only money but also other variables related to the social situation. A different approach is to drop the "bargaining theory axioms" (cf. Section 1) and define a non-welfaristic solution directly on the class of distribution problems, instead of on a class of derived bargaining problems. The advantage of this approach is that the distributive principles the bargainers are assumed to apply can be formulated in terms of amounts of goods to be distributed. Such principles then may depend on economic and ethical aspects of the situation.

References


