Stochastic implications of the life cycle consumption model under rational habit formation

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1 Introduction

Hall [1978] showed that for a time-separable utility function the life cycle consumption model implies a first order autoregressive process for marginal utility of consumption. Empirical analyses typically lead to rejection of Hall's conjecture. One explanation put forward in the literature concerns the assumption of a representative consumer. Macroeconomic data violate the martingale property as a result of aggregation issues (Clarida [1991]). Within the framework of a representative consumer rejection implies a departure from the conventional assumptions. The presence of credit constraints, leading to a reduction of the planning horizon of the consumer, is an example (see Mariger [1987]). A bounded-rationality model in which the consumer voluntarily reduces the planning horizon, is presented in Winder and Palm [1989]. Alternatively, the assumption of an intertemporally separable utility function may be violated. This latter case will be re-examined in the present paper.

This paper extends Hall's result by showing that for an exponential utility function an appropriate pattern of nonseparability over time of preferences will lead to an arbitrary autoregressive integrated moving average (ARIMA) process for consumption. The result of this paper suggests that ignoring for instance habits may explain the frequent rejection of the life cycle hypothesis. It also suggests that the significance of lagged consumption in a univariate model for the variation of consumption should not lead to rejecting the life cycle model per se but could lead to disregarding the assumption of time-separable preferences.

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The present paper belongs to the growing strand of literature on nonseparabilities over time of preferences which result from habit persistence (see e.g. Pollak [1970], Constantinides [1990]), rational addiction (see e.g. Stigler and Becker [1977] and Becker and Murphy [1988]), adjustment costs as suggested by Houthakker and Taylor [1970], durability of consumption goods (see e.g. Eichenbaum and Hansen [1990]) and habit persistence or durability (see e.g. Ferson and Constantinides [1991], Braun, Constantinides and Ferson [1998]).

The paper is organised as follows. In section 2, we first derive the main result under the assumption that the consumer has point expectations with respect to future labour income. We give the closed-form solution for consumption expressed as an ARIMA model with innovations which are proportional to the innovations of the process for labour income. Second, the model in which the consumer is assumed to maximise expected utility is discussed. Section 3 concludes the study.

2 Theory

In this section we discuss the implications of the life cycle consumption model when preferences at time $t$ depend on a finite number of past realisations of consumption. At the first instance we assume that the consumer has point expectations with respect to future labour income, e.g. $E(y_{t+i} \mid I_t)$. $E$ denotes the expectations operator, and $I_t$ is the set of information available at time $t$. It is assumed that the consumer knows the value of $y_t$ when taking a decision about $c_t$. Hence, $E(y_t \mid I_t) = y_t$. The consumption plan for period $t+i$ made at time $t$ is denoted as $c_{t+i}$. Obviously, for period $t$ we have $c_t = c_t$ as the realisation. At time $t$ the representative consumer is assumed to maximise his life time utility subject to the life time budget constraint under point expectations of income

$$\max \sum_{i=0}^{\infty} \beta^i U (\Phi(L)c_{t+i})$$  \hspace{1cm} (1)

s.t. $\sum_{i=0}^{\infty} (1+r)^{-i} c_{t+i} = (1+r)c_{t-1} + \sum_{i=0}^{\infty} (1+r)^{-i} E(y_{t+i} \mid I_t)$,

with $U' > 0$ and $U'' < 0$, where $U'$ and $U''$ are the first and second derivatives of $U$, $c_{t-1}$ is real financial wealth, $\beta$ is the time preference parameter ($0 < \beta < 1$), $r$ is the real rate of interest, which is assumed to be constant ($0 < r < 1$) and $\Phi(L)$ is a polynomial of order $p$ in the lag operator $L$,

$$\Phi(L) = 1 - \phi_1 L - \cdots - \phi_p L^p,$$
with factorisation
\[ \Phi(L) = (1 - \pi_1 L)(1 - \pi_2 L) \cdots (1 - \pi_p L). \]

In the subsequent analysis it will follow that we have to impose the condition that the roots of \( \Phi(L) = 0 \) must lie on or outside the unit circle, that is \( |\pi_i| \leq 1, i = 1, \ldots, p \). The life time budget constraint in (1) results from successive substitution of the period by period budget constraints
\[ a_{t+i} = (1 + r)a_{t+i-1} + E(y_{t+i} | I_t) - c^t_{t+i}, \quad i = 0, 1, \ldots \]
and the boundary condition
\[ \lim_{t \to \infty} (1 + r)^{-t} a_{t+i} = 0, \]
which is the transversality condition. The life time budget constraint as formulated in (1) is meaningless, unless the infinite sums converge. This leads to the requirement that \( c^t_{t+i} \) and \( E(y_{t+i} | I_t) \) are of exponential order less than \( (1 + r) \). A sequence \( z_{t+i} \) is of exponential order less than \( (1 + r) \), when there exist \( z_0 \) and \( K > 0 \) such that for every \( i > z_0, | z_{t+i} | < K x^i \) for some \( x \in (1, 1 + r) \). It is also required that the life cycle utility determined by (1) does not diverge to infinity. When \( U \) is bounded from above, convergence of the target function is guaranteed.

The first order conditions of (1) consist of a system of difference equations
\[
\frac{\partial}{\partial c^t_{t+\ell}} \left[ \sum_{i=t}^{t+p} \beta^i U(\Phi(L)c^t_{t+i}) \right] = \frac{1}{1 + r} \frac{\partial}{\partial c^t_{t+\ell-1}} \left[ \sum_{i=t-1}^{t+p-1} \beta^i U(\Phi(L)c^t_{t+i}) \right], \quad \ell = 1, 2, \ldots \quad (2)
\]
Determining the solution of (1) at time \( t \) corresponds to solving the \((p+1)\) th order difference equation (2) subject to the \( p \) initial conditions \( c_{t-1}, c_{t-2}, \ldots, c_{t-p} \) and the budget constraint in (1).

Since
\[
\frac{\partial}{\partial c^t_{t+\ell}} \left[ \sum_{i=t}^{t+p} \beta^i U(\Phi(L)c^t_{t+i}) \right] = \sum_{i=t}^{t+p} \beta^i U'(\Phi(L)c^t_{t+i}) \left( \frac{\partial(\Phi(L)c^t_{t+i})}{\partial c^t_{t+\ell}} \right) = \sum_{i=t}^{t+p} \beta^i U'(\Phi(L)c^t_{t+i}) \frac{\partial(\Phi(L)c^t_{t+i})}{\partial c^t_{t+\ell}}
\]

(1) In model (1) the current decision \( c_t \) is affected by past choices of consumption. Winder and Palm (1989) examine the model with finite time horizon without habits, that is \( \Phi(L) = 1 \). Muehlbauer (1988) discusses the version of model (1) in which expected utility is maximised for the case that \( \Phi(L) \) is of order 1.
for all \( \ell \), (2) can be reformulated as

\[
\sum_{i=\ell}^{\ell+p} \beta^i U'(\Phi(L)c^i_{t+i}) \frac{\partial(\Phi(L)c^i_{t+i})}{\partial c^i_{t+\ell}} = \frac{1}{1+r} \sum_{i=\ell}^{\ell+p} \beta^i U'(\Phi(L)c^i_{t+i}) \frac{\partial(\Phi(L)c^i_{t+i})}{\partial c^i_{t+\ell-1}} = \frac{1}{\beta(1+r)} \sum_{i=\ell}^{\ell+p} \beta^i U'(\Phi(L)c^i_{t+i-1}) \frac{\partial(\Phi(L)c^i_{t+i-1})}{\partial c^i_{t+\ell-1}}.
\]

Using \( \frac{\partial(\Phi(L)c^i_{t+i-1})}{\partial c^i_{t+\ell}} = \frac{\partial(\Phi(L)c^i_{t+i-1})}{\partial c^i_{t+\ell}} \), \( i = \ell, \ldots, \ell+p \) for all \( \ell \), it follows that

\[
\sum_{i=\ell}^{\ell+p} \beta^i \left[ U'(\Phi(L)c^i_{t+i}) \frac{1}{\beta(1+r)} U'(\Phi(L)c^i_{t+i-1}) \right] \times \frac{\partial(\Phi(L)c^i_{t+i})}{\partial c^i_{t+\ell}} = 0 \quad , \quad \ell = 1, 2, \ldots
\]

A sufficient condition for (3) to hold true is

\[
U'(\Phi(L)c^i_{t+i}) = \beta^i(1+r)^{-1} U'(\Phi(L)c^i_{t+i-1}) \quad , \quad i = 1, 2, \ldots
\]

**Theorem 1** With \( U(c) = -\gamma^{-1} \exp(-\gamma c) \), \( \gamma > 0 \), the optimisation problem (1) implies\(^{(2)}\)

\[
\Phi(L) \Delta c_{t+1} = \gamma^{-1} \ln [\beta(1+r)] + \epsilon_{t+1}
\]

with the innovation \( \epsilon_{t+1} \equiv c_{t+1} - c^i_{t+1} = [r/(1+r)]w_{t+1} \), where

\[
w_{t+1} = \sum_{t=0}^{\infty} (1+r)^{-1} \left[ E(\Phi(L)y_{t+1+1} | I_{t+1}) - E(\Phi(L)y_{t+1+1} | I_t) \right].
\]

**Proof:** For the exponential utility function \( U \), the optimal consumption plan for period \( t+i \) made at time \( t \), \( c^i_{t+i} \) corresponds to the solution of

\(^{(2)}\) For a comprehensive discussion of the properties of the exponential utility function, see e.g. Kimball (1980), Blanchard and Mankiw (1988). The exponential utility function, which under uncertainty has constant absolute risk aversion and constant absolute prudence measure (defined as \(-U''/U''\), with \(U''\) being the third derivative of \(U\)) has been used by Caballero (1990, 1991). It has the advantage to allow for a closed form solution. This advantage outweighs the fact that marginal utility associated with an exponential functional form is finite even at zero consumption.
the linear difference equation of order \((n + 1)\)

\[
\Phi(L)(1 - L)c^*_i = \gamma^{-1} \ln[\beta(1 + r)], \quad i = 1, 2, \ldots
\]  

(7)

with initial conditions \(c_{t-1}, c_{t-2}, \ldots, c_{t-n}\) and the life time budget constraint (1). The literature on linear difference equations (see e.g. Sargent (1979)) provides us immediately with the form of the solution of (7). It can easily be checked that to avoid a contradiction with the requirement that \(c^*_i\) is of exponential order less than \(1 + r\) for all \(r > 0\), it is necessary to impose the restrictions \(|\pi_t| \leq 1, \ i = 1, \ldots, n\).

We proceed by defining auxiliary variables \(c^*_i = \Phi(L)c^*_{i+1}, i = 0, 1, \ldots\). Solving the difference equation (7) subject to the \(p\) initial conditions and the budget constraint is equivalent to solving the linear first order difference equation

\[
c^*_i = c^*_i + \gamma^{-1} \ln[\beta(1 + r)], \quad i = 1, 2, \ldots
\]  

(8)

with one boundary condition, namely the life time budget constraint expressed in terms of \(c^*_1\)

\[
\sum_{i=0}^{\infty}(1 + r)^{-i}c^*_i = (1 + r)\Phi(L)u_{t-1} + \sum_{i=0}^{\infty}(1 + r)^{-i}E\Phi(L)y_{t+i} | I_t) .
\]  

(9)

Expression (8) can be written as

\[
c^*_i = c^*_i + i\gamma^{-1} \ln[\beta(1 + r)], \quad i = 1, 2, \ldots
\]  

(10)

because \(c^*_i = c^*_i\), and substitution into the life time budget constraint (9) yields

\[
c^*_1 = \frac{1 + r}{\gamma^{-1} \ln[\beta(1 + r)]} \frac{1 + r}{\gamma^2}
\]

\[
= (1 + r)\Phi(L)u_{t-1} + \sum_{i=0}^{\infty}(1 + r)^{-i}E\Phi(L)y_{t+i} | I_t) .
\]  

(11)

Carrying out the same operations for period \(t + 1\) leads to

\[
c^*_t = \frac{1 + r}{\gamma^{-1} \ln[\beta(1 + r)]} \frac{1 + r}{\gamma^2}
\]

\[
= (1 + r)\Phi(L)u_1 + \sum_{i=0}^{\infty}(1 + r)^{-i}E\Phi(L)y_{t+i} | I_{t+1}) .
\]  

(12)

Dividing (12) by \(1 + r\), substituting \(u_t = (1 + r)u_{t-1} + y_t - c_t\) and subtracting (11) yields

\[
c^*_t = \frac{1}{\gamma^{-1} \ln[\beta(1 + r)]} + \frac{r}{1 + r}u_{t+1} .
\]
where \( w_{t+1} \) has been defined in (6). When we substitute \((1-L)c_{t+1} = \Phi(L)\Delta c_{t+1} \) we get the autoregressive process for \( \Delta c_{t+1} \) given in (5). □

Consumption follows an autoregressive integrated (ARI) stochastic process. As unit roots are permitted the order of integration may be larger than one for an appropriate choice of the lag polynomial \( \Phi(L) \) (obviously, integration of order zero is excluded). Notice that when the process for labour income is specified, e.g. when income is generated by an ARIMA process it is straightforward to show that the consumption innovation is a linear transformation of the income innovation. Whether or not there will be excess sensitivity of consumption to income depends on the form of the income process and the values of \( r \) and the \( \phi_i \)'s.

Pollak [1970] mentions the possibility of infinite memory which can be modelled by assuming \( \Phi(L) \) to be a rational polynomial \( \Phi(L) = \phi(L)\theta^{-1}(L) \), where \( \phi(L) \) and \( \theta(L) \) are finite degree polynomials in \( L \). Then, the stochastic process for consumption is an arbitrary ARIMA process

\[
\phi(L)\Delta c_{t+1} = \theta(L)\gamma^{-1}\ln[\beta(1+r)] + \theta(L)e_{t+1}.
\] (13)

The model with infinite memory may be used to describe consumption behaviour with respect to durable goods with an infinite life time studied by for instance Mankiw [1982]. When we assume that the stock of durable goods \( K_{t+1} \) evolves according to \( K_{t+1} = (1-\delta)K_{t+1} + c_{t+1} \), we have for the service flow \( s_{t+1} = \theta K_{t+1} = (1-(1-\delta)L)^{-1}\theta c_{t+1} \). Hence, \( s_{t+1} \) depends on durable goods acquired far back into the past. Given an intertemporally additive utility function with arguments \( s_{t+1} \), the resulting model is a special case of (13). Consumption should obey an IMA(1,1) process, in which the moving average parameter is related to the rate of depreciation. This result was derived before by Mankiw [1982]. Notice that the set-up described in our paper opens up the possibility to handle more general schemes of depreciation and alternative preference structures or to combine habit persistence and durability (see e.g. Persson and Constantinides [1991, 1993]).

Finally, we consider the model in which the consumer maximises expected life time utility

\[
\max E_t \left[ \sum_{t=0}^{\infty} \beta^t U(\Phi(L)c_{t+1}) \right]
\] (14)

subject to the budget constraint

\[
\sum_{i=0}^{\infty} (1+r)^{-i} c_{t+i} = (1+r)c_{t-1} + \sum_{i=0}^{\infty} (1+r)^{-i} y_{t+i}.
\] (15)
Theorem 2 With \( U(c) = -\gamma^{-1} \exp(-\gamma c) \), \( \gamma > 0 \), and under the assumption of normality of consumption, the optimisation problem (14)-(15) implies

\[
\Phi(L) \Delta c_{t+1} = \gamma^{-1} \ln [\beta(1 + r)] + \frac{1}{2} \gamma \sigma^2 (\Phi(L)c_{t+1}) + u_{t+1}
\]

(16)

with \( u_{t+1} = c_{t+1} - E_t(c_{t+1}) \).

Proof: See Appendix.

Since the conditional variance of a normally distributed variable does not depend on the realisations of the variables in the conditioning set, the model (16) for consumption is observationally equivalent to (5), although its drift parameter differs from that in (5). Of course, under a policy intervention leading to a structural change in the variance of \( c_{t+1} \), the two models behave differently. Finally as a result of the constancy of the absolute prudence coefficient implied by the exponential utility function, increasing income uncertainty does not increase the marginal propensity to consume. An increase of income uncertainty shifts the consumption function but does not modify its slope. A variation of current income has the same impact on current consumption as under certainty equivalence: it depends on the way this variation changes income expectations, and therefore on the process for income (see also Blanchard and Mankiw [1988], Kimball [1990], Zeldes [1989]).

3 Concluding remarks

In this paper we considered the life cycle model under rational habit formation. It was shown that for the exponential utility function with an appropriate pattern of rational habits an arbitrary ARIMA process is obtained as the decision rule for consumption. An implication of this finding is that checking the significance of past realisations of consumption is not so much a test of the life cycle model as a test of the specific form of rational habit formation. Obviously, the results suggest that ignoring habits or other forms of nonseparability may explain the frequent rejection of the life cycle hypothesis.

Our approach based on internal habit formation differs from that by Campbell and Cochrane [1994] who assume consumption growth as an i.i.d. log-normal process and an (external) individual's habit level that depends on the history of aggregate consumption rather than the individual's own past consumption. In their model, habits adapt non-linearly to the history of consumption in a way that keeps habits always below consumption and keeps marginal utility always finite and positive.
While their specification simplifies the analysis and helps to reconcile a constant riskless interest rate with a random walk consumption process, it does not account for serial correlation in consumption growth rates.

**APPENDIX**

The expected utility case

In the model (14)-(15) the life time budget constraint results from successive substitution of the period-by-period budget constraint

\[ a_{i+i} = (1 + r)a_{i+i-1} + y_{i+i} - c_{i+i} \quad , \quad i = 0, 1, 2, \ldots \quad \text{(A.1)} \]

We reformulate the maximisation problem in terms of the transformed variables \( c_{i+i} = \Phi(L)c_{i+i} \). Since \( \Phi(L) \) is a linear operator, the system of period-by-period constraints is equivalent to the following system expressed in terms of the transformed variables \( c_{i+i}^*, a_{i+i}^* = \Phi(L)a_{i+i} \) and \( y_{i+i}^* = \Phi(L)y_{i+i} \)

\[ a_{i+i}^* = (1 + r)a_{i+i-1} + y_{i+i}^* - c_{i+i}^* \quad , \quad i = 0, 1, 2, \ldots \quad \text{(A.2)} \]

In terms of the transformed variables the maximisation problem reads as

\[ \max E_t \left\{ \sum_{i=0}^{\infty} \beta^i U(c_{i+i}^*) \right\} \quad \text{(A.3)} \]

S.T. \( a_{i+i}^* = (1 + r)a_{i+i-1} + y_{i+i}^* - c_{i+i}^* \quad , \quad i = 0, 1, 2, \ldots \)

Problem (A.3) corresponds with the standard formulation studied by Hall [1978]. Since \( \Phi(L) \) is a linear operator, maximisation of the objective function of (A.3) with respect to \( c_{i+i}^* \) yields the same system of first order conditions as maximisation with respect to \( c_{i+i} \). The first order conditions imply

\[ E_t \left\{ U'(c_{i+i}^*) \right\} = [\beta(1 + r)]^{-i} U'(c_i^*) \quad , \quad i = 1, 2, \ldots \quad \text{(A.4)} \]

For \( i = 1 \) we obtain after substituting \( c_{i+j}^* = \Phi(L)c_{i+j} \), \( j = 0, 1 \).

\[ E_t \left\{ U'(\Phi(L)c_{i+1}) \right\} = [\beta(1 + r)]^{-1} U'(\Phi(L)c_0) \quad \text{(A.5)} \]

which specialises for the exponential utility function to

\[ E_t \left\{ \exp(-\gamma \Phi(L)c_{i+1}) \right\} = [\beta(1 + r)]^{-1} \exp(-\gamma \Phi(L)c_0). \quad \text{(A.6)} \]
Without additional assumptions it is not possible to find an explicit solution for the consumption decision $c_t$. Following Palm and Winder [1990] we assume therefore that consumption is normally distributed. In that case (A.6) can be rewritten as

$$E_t \{ \Phi(L) \Delta c_{t+1} \} = \frac{1}{\gamma} \ln [\beta(1 + r)] + \frac{1}{2} \gamma \sigma^2 \{ \Phi(L)c_{t+1} \}.$$  \hspace{1cm} (A.7)

Defining the consumption innovation $u_{t+1} = c_{t+1} - E_t(c_{t+1})$ we obtain (16). Since consumption is normally distributed, the conditional variance does not depend on past realisations of consumption. Model (A.7) is therefore observationally equivalent to (5) as long as the second r.h.s. term of (A.7) is constant. In order to express the consumption innovation $u_{t+1}$ in terms of characteristics of the income process, one needs the closed form solutions for the consumption levels chosen in period $t$ and $t+1$. For the model in which the expected life time utility is maximised the expressions become quite awkward. For an illustration we refer to Palm and Winder [1990], who investigate the model without habit formation. In the model discussed in this annex one can show that the consumption innovation is a linear transformation of the income innovation, whereby the proportionality factor does not only depend on the level of the real interest rate and the parameters of the income process, but also on the parameters of $\Phi(L)$. This result is similar to that of the model (5) derived under the assumption of point expectations. □

REFERENCES


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