Consistent restricted Shapley values

Jean Derks\textsuperscript{a}, Hans Peters\textsuperscript{b,*}

\textsuperscript{a}Department of Mathematics, University of Maastricht, PO Box 616, 6200 MD Maastricht, The Netherlands

\textsuperscript{b}Department of Economics, University of Maastricht, PO Box 616, 6200 MD Maastricht, The Netherlands

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Abstract

Transferable utility games with an additional power structure on the coalitions are considered. This power structure is not given explicitly, but only implicitly via a value; a value is a map that assigns an $N$-vector to every game with player set $N$. The implicit power structure is described by the concept of effectiveness of a coalition for a given value. The effectiveness of coalitions is constrained by axioms; in particular, the collection of effective coalitions is assumed to be closed under taking unions. Other axioms concern efficiency and consistency in a sense related to the consistency axiom of Hart and Mas-Colell. The main result of the paper is an axiomatic characterization of a class of restricted Shapley values, with the effective coalitions forming a lattice.

Keywords: Transferable utility game; Power structure; Effectiveness; Axiom system; Consistency

1. Introduction

Consider the situation where a number of firms are connected in a network, such as a network of cables connecting the firms with a power supplier (or a network of pipelines connecting the firms with an oil reservoir, or an infrastructure of roads or railways connecting the firms with a harbour). By cooperating, coalitions of firms may make additional profits – or save costs – but only if all firms in the coalition can be connected with the 'power supplier'. Observe that such a situation is characterized by two structures. First, there is a function that
assigns to each coalition its potential profits, i.e. profits that can be realized, given that all firms in the coalition can be connected with the power supplier. Secondly, there is a structure – described, for instance, by a graph or network – that determines which coalitions can actually realize those profits. While a situation such as this may often be described by just one characteristic function, by incorporating in some way the coalitions that are capable of realizing their profits into the function that describes those profits, it is not obvious that this is the best way of modelling the situation. In particular, the infrastructure (of cables, pipelines, roads or railways) may be more stable than the potential profits, which may depend on prices in input and output markets, and other unstable factors.

To be more specific, we consider the following numerical example. There are four firms and one power station connected in a network as follows: firm 1 and firm 4 are both connected with the power station, and also with firm 2; and firm 3 is only connected with firm 2. The potential profits of the coalitions, which can be realized only if the firms can be connected with the power station, are as follows. One-firm coalitions make profits of 1, two-firm coalitions profits of 3, three-firm coalitions profits of 4, and the grand coalition of all four firms makes a profit of 7; by convention, the empty coalition makes zero profits. The coalitions \( \{2\}, \{3\} \) and \( \{2, 3\} \), however, are not connected with the power station, so make the same profits as the empty coalition, i.e. zero. The coalition \( \{1, 3\} \) makes the same profit as the coalition \( \{1\} \), since firm 3 cannot be connected. Similarly, \( \{3, 4\} \) makes the same profits as \( \{4\} \), and \( \{1, 3, 4\} \) the same as \( \{1, 4\} \). If we assume that the grand coalition forms and we apply a solution concept such as the Shapley value (Shapley, 1953; in this paper, the focus is on the Shapley value) to this modified profits game, we obtain the pay-off vector \( (25, 23, 11, 25) \). This clearly reflects the structure of the network; without this additional structure, the Shapley value would just be the vector \( (21, 21, 21, 21) \). Note that, if the profits were to change but the network remained unaltered, then a similar qualitative effect of the network structure on the Shapley value would be obtained. Therefore, it seems natural to model the network structure by means of the value; which is what we will do in this paper.

As a second example, suppose that the set of players is a parliament, partitioned in political parties. Suppose that each political party is subject to the strict party discipline that only unanimous votes may be cast. In this case, only the cooperation of complete political parties (fractions) is useful, because individual members are not expected to vote on their own. In a similar formal model, one may think of the players as divisions partitioned into firms, which may only cooperate – for instance, exchange technological information – with the consent of the other divisions within the same firm.

More formally, we try to capture this idea of a fixed power structure in the framework of games with transferable utility. We assume that this power structure is reflected by a value, i.e. a map that assigns to every game with transferable
utility a distribution among the players. More precisely, we will assume that every coalition is either effective, i.e. affects the distribution in every game where it is present, or ineffective, i.e. never affects the distribution. Consequently, the effectiveness of a coalition is reflected by the value and does not depend on the particular game (characteristic function) at hand. As a minimal condition, we assume that the union of effective coalitions is again effective – as is the case in the examples above. Both assumptions are formalized by axioms.

We will propose three other axioms. The first of these is an axiom of efficiency. The second axiom requires the value to be consistent across games with varying numbers of players, and is closely related to the consistency requirement of Hart and Mas-Colell (1989). Finally, there is an axiom that first determines the value for players who belong to zero or one effective coalitions and for who all other coalitions are ineffective, and then establishes the basic distribution between players in two disjoint minimal effective coalitions. The main result of the paper is that these axioms determine a class of coaltionally restricted Shapley values. A consequence of the axioms turns out to be that the intersection of effective coalitions is again effective, i.e. the class of effective coalitions forms a lattice. In the second example above, the effective coalitions form a lattice but, in the first example, they clearly do not.

Our main result generalizes the Hart-Mas-Colell characterization of the Shapley value. The uniqueness part of the proof also implies a new proof of the uniqueness part of the Hart-Mas-Colell theorem, while the necessity part uses the fact, proved by Hart and Mas-Colell, that the Shapley value is consistent. A completely self-contained proof can be based on the concept of a restricted potential – generalizing the Hart-Mas-Colell potential – at the cost of brevity (see Derks and Peters, 1995).

Coalitionally restricted Shapley values are obtained by considering only dividends – or marginal contributions in the probabilistic representation – of effective coalitions. The characterized class forms a subclass of the class of restricted Shapley values introduced in Derks and Peters (1993). Recently, restrictions on coalitions in games have been discussed by several other authors. In fact, the paper by Derks and Peters (1993) arose as an attempt to generalize some of these approaches. It presents an axiomatic characterization of a class of restricted Shapley values for which the set of effective coalitions is closed under taking unions, and not necessarily under taking intersections. This characterization is based on an axiom of monotonicity related to the condition used by Young (1985) to characterize the Shapley value. Related works are Gilles and Owen (1991), Gilles et al. (1992), Faigle and Kern (1992), Hsiao and Raghavan (1990) and Feldman (1993). Earlier contributions include Myerson (1977) and Owen (1986). The lattice structure in the context of cooperative game theory is also studied by Derks and Gilles (1995) and by Gilboa and Lehrer (1991).

The organization of the paper is as follows. In Section 2, after some preliminary
definitions, we introduce the axioms briefly described above. In Section 3, we introduce restricted Shapley values, by means of dividends, and we also present a corresponding probabilistic formula. Next, some preparations are made for the main result, which is stated and proved in Section 4. Section 5 concludes with some remarks on other characterizations.

2. Preliminaries and axioms

Let $Z \subseteq \mathbb{N}$ denote the set of potential players (where $\mathbb{N}$ is the set of natural numbers, $\subseteq$ denotes inclusion and $\subset$ denotes strict inclusion). The reason for considering a (possibly large) set of potential players is that we wish to be able to vary the number of players. A coalition is a finite subset of $Z$. For one-person coalitions, we often omit braces, i.e. we write $i$ instead of $\{i\}$.

A (transferable utility) game is a pair $(N, v)$, where $N$ is a coalition and $v : 2^N \rightarrow \mathbb{R}$ is a function with $v(\emptyset) = 0$. The function $v$ is the characteristic function. The number $v(S)$ is called the worth of coalition $S$. By $\mathcal{G}$, we denote the set of all games. For $(N, v) \in \mathcal{G}$ and $M \subseteq N$, the characteristic function of the game $(M, v) \in \mathcal{G}$ is the map $v$ restricted to $2^M$. A value is a map $\psi$ that assigns a vector in $\mathbb{R}^N$ to every game $(N, v) \in \mathcal{G}$, to be interpreted as a solution of the game. A well-known value is the Shapley value (Shapley, 1953, Section 3).

For a given value $\psi$, a non-empty coalition $S$ is called $\psi$-effective if, for all games $(N, v), (N, w) \in \mathcal{G}$ with $S \subseteq N$, $v(T) = w(T)$ for all $T \subseteq N$ with $T \neq S$, and with $v(S) \neq w(S)$, we have $\psi(N, v) \neq \psi(N, w)$. A coalition $S$ is called $\psi$-ineffective if for all games $(N, v), (N, w) \in \mathcal{G}$ with $S \subseteq N$ and $v(T) = w(T)$ for all $T \subseteq N$ with $T \neq S$, we have $\psi(N, v) = \psi(N, w)$. Observe that the empty coalition is $\psi$-ineffective. If it is clear which value is meant, then we will often use the expressions effective and ineffective, omitting the prefix $\psi$. A coalition $S$ is a maximal $\psi$-effective coalition in $N$ if it is $\psi$-effective, $S \subseteq N$ and every $S \subset T \subseteq N$ is not $\psi$-effective. For $k \in \mathbb{N} \cup \{0\}$, a player in $N$ is called a $k$-player (in $N$) if, of all coalitions in $N$ containing that player, exactly $k$ coalitions are effective, while the other coalitions are ineffective. Note that this concept also depends on a given value $\psi$.

Observe that the classification in effective and ineffective coalitions is not exhaustive; a coalition might be influential and affect the value in one game (or, more precisely, pair of games), but not exert any influence in another game. Such an occurrence, however, would be inconsistent with the assumption of a fixed power structure, independent of the particular game at hand. This consideration is formalized by our first axiom. (Axioms are formulated for an arbitrary value $\psi$.)

**Axiom 1.** Every non-empty coalition, $S$ is either $\psi$-effective or $\psi$-ineffective.
Also, the second axiom concerns the power structure among coalitions. It requires the collection of effective coalitions to be closed under taking unions.

Axiom 2. For all coalitions $S$ and $T$, if $S$ and $T$ are $\psi$-effective, then so is $S \cup T$.

We will assume that, in every game $(N, v)$, some maximally $\psi$-effective coalition is formed and that the worth of this coalition is distributed among the players of $N$. This is formalized by the following efficiency requirement.

Axiom 3. For all $(N, v) \in \mathcal{G}$ such that $N$ contains a $\psi$-effective coalition, we have
\[
\sum_{i \in N} \psi_i(N, v) = v(S)
\]
for some maximal $\psi$-effective coalition $S$ in $N$.

Obviously, under Axiom 2, such a maximal $\psi$-effective coalition is unique.

The next axiom is a consistency property closely related to that introduced by Hart and Mas-Colell (1989) for the Shapley value. Let $(N, v) \in \mathcal{G}$, and $U \subseteq N$. The reduced game $(N \setminus U, v_{\psi,U})$ is defined as follows. For every $\emptyset \neq S \subseteq N \setminus U$, we have
\[
v_{\psi,U}(S) = \sum_{i \in S} \psi_i(S \cup U, v).
\]
Thus, in the reduced game, the worth of coalition $S$ is equal to what the players in $S$ receive together if they were to play the (restriction of the ) original game with the coalition $U$. If $\psi$ is efficient in the usual sense, implying that $\sum_{i \in S \cup U} \psi_i(S \cup U, v) = v(S \cup U)$, then the reduced game can be written as
\[
v_{\psi,U}(S) = v(S \cup U) - \sum_{i \in U} \psi_i(S \cup U, v)
\]
which is the definition of the Hart–Mas-Colell reduced game.

The following axiom of consistency (or reduced game property) requires that, in a reduced game for an effective coalition, the players do not redistribute the total of what they receive in the original game. In this sense, the value is consistent in the distributions if the number of players varies.

Axiom 4. For every $(N, v) \in \mathcal{G}$ and every $U \subseteq N$ with $N \setminus U$ $\psi$-effective, we have
\[
\psi_i(N \setminus U, v_{\psi,U}) = \psi_i(N, v)
\]
for every $i \in N \setminus U$. 
It might not be obvious that consistency is a natural requirement in our framework with an additional power structure, since this structure could be torn up by considering reduced games for arbitrary smaller coalitions. For this reason, in Axiom 4, only reduced games for effective coalitions are considered. In this way, we avoid, for instance, players becoming 0-players in a reduced game, who were not 0-players in the original game (cf. Lemma 4). Actually, it will follow from our main result that, if a value satisfies Axioms 1–4, and Axiom 5 below, then the corresponding effective coalitions form a lattice. This implies that, in the reduced game, the original power structure is preserved.

The consistency axiom will be used to relate distributions assigned by \( \psi \) across games with distinct numbers of effective coalitions. To apply this procedure, we need to assume a few basic distribution codes. This is done in our final axiom, which may be regarded as an extension of the standardness axiom of Hart and Mas-Colell (1989).

**Axiom 5.** Let \( S \) and \( T \) be coalitions.

(a) If \( T \subseteq S \) and \( T \) consists exactly of the 0-players in \( S \), then \( \psi_i(S, v) = 0 \) for every \( (S, v) \in \mathcal{G} \) and \( i \in T \).

(b) If \( S \) is \( \psi \)-effective, \( T \subseteq S \) and \( T \) consists exactly of the 1-players in \( S \), then
\[
\psi_i(S, v) = \frac{1}{|S|} [v(S) - v(S \setminus T)]
\]
for every \( (S, v) \in \mathcal{G} \) and \( i \in T \).

(c) If \( N = S \cup T \) with \( S \cap T = \emptyset \), and such that \( N, S, \) and \( T \) are the only \( \psi \)-effective coalitions in \( N \), then
\[
\psi_i(N, v) = (1/|T|)v(T) + (1/|N|)[v(N) - v(S) - v(T)]
\]
for every \( (N, v) \in \mathcal{G} \) and \( i \in T \).

The cases (a), (b) and (c) of Axiom 5 take care of 0-players, 1-players and 2-players respectively. A 0-player obtains 0. A 1-player obtains his/her fraction of the marginal contribution to the grand coalition; observe that, since, in the formulation of this axiom, the grand coalition is effective by assumption, it is the unique effective coalition to which the members of \( T \) belong. In (c), the basic divisional proportion between players in two minimal disjoint effective coalitions is determined.

Note that Axiom 5 is a generalization of the standardness axiom of Hart and Mas-Colell, as formulated for the Shapley value; in that case, all coalitions are effective, implying that only part (c) of Axiom 5 applies, and only for the case of \( S = \{j\}, T = \{i\} \). Then, \( \psi_i(N, v) = v(i) + \frac{1}{2} [v(i, j) - v(i) - v(j)] \), which is the requirement of the standardness axiom.

The main result of the paper is the characterization of a class of coalitionally restricted Shapley values by Axioms 1–5. Before we can formulate and prove this result, we first need to introduce and study these solutions; this is the subject of the next section.
3. Restricted Shapley values

As in the preceding section let \( Z \) be the set of potential players. Let \( \Omega \) be a (finite or infinite, possibly empty) collection of coalitions which is closed under taking unions. Such a collection is called a power structure. An element \( S \) of \( \Omega \) is called minimal if it contains no proper subset which is in \( \Omega \). The collection \( \Omega \) is a lattice if, for every pair of its elements with non-empty intersection, this intersection is in \( \Omega \). For an arbitrary coalition \( S \), let

\[
S = \bigcup_{T : T \subseteq S, T \in \Omega} T
\]

where the union over an empty collection is, by definition, the empty set. Note that, since \( \Omega \) is union-closed by definition, if \( S \neq \emptyset \), then \( S \) is the (unique) maximal subset of \( S \) which is in \( \Omega \). Thus, for \( S \neq \emptyset \), \( \bar{S} = S \) if and only if \( S \in \Omega \).

For \( (N, v) \in \mathcal{G} \) and a power structure \( \Omega \), we define dividends \( \Delta^\Omega_v(S) \) for coalitions \( S \in \Omega \) recursively, as follows:

\[
\Delta^\Omega_v(S) = \begin{cases} 
v(S), & \text{if } S \text{ is a minimal element of } \Omega \\ v(S) - \sum_{T : T \subseteq S, T \in \Omega} \Delta^\Omega_v(T), & \text{otherwise} \end{cases}
\]

Then, a value \( \varphi^\Omega \) on \( \mathcal{G} \) is defined as follows. For every \( (N, v) \in \mathcal{G} \) and every \( i \in N \), we have

\[
\varphi^\Omega_i(N, v) = \sum_{S : i \subseteq N, S \in \Omega} \frac{\Delta^\Omega_v(S)}{|S|}
\]

where summation over the empty set yields zero, by definition. If \( \Omega \) is the lattice of all non-empty coalitions, then \( \varphi^\Omega \) is the Shapley value, denoted by \( \varphi \) (see Shapley, 1953; Harsanyi, 1963). In general, we call \( \varphi^\Omega \) a restricted Shapley value.

Restricted Shapley values were introduced earlier in Derks and Peters (1993) by their probabilistic formula. The equivalence of both formulations is proved in Section 3 of Derks and Peters (1993), and stated next as a lemma. For a coalition \( S \), \(|S|\) denotes the cardinality of \( S \).

**Lemma 1.** Let \( \Omega \) be a power structure. Then, for every \( (N, v) \in \mathcal{G} \) and every \( i \in N \), we have

\[
\varphi^\Omega_i(N, v) = \frac{|S|!(|N| - |S| - 1)!}{|N|!} \left[ v(S \cup \{i\}) - v(S) \right]
\]

For the Shapley value, where \( \Omega \) consists of all non-empty coalitions, Lemma 1 just gives the familiar equivalence between Shapley's probabilistic formula and the Harsanyi dividends.

Note that, from the definition of \( \varphi^\Omega \) or from Lemma 1,\(^1\) we have

\[^1\] The idea to consider the game \( \tilde{\mathcal{G}} \) was suggested by a referee.
\[ \phi^\Omega(v) = \phi(\bar{v}), \quad \text{where } \bar{v}(S) := v(\bar{S}) \text{ for all } S \subseteq N \tag{1} \]

The following lemma will imply the efficiency (Axiom 3) of restricted Shapley values.

**Lemma 2.** Let \( \Omega \) be a power structure and let \((N, v) \in \mathcal{G} \). Then, we have

\[ \sum_{i \in N} \phi_i^\Omega(N, v) = v(\bar{N}) \]

**Proof.** We have

\[ \sum_{i \in N} \phi_i^\Omega(N, v) = \sum_{i \in N} \phi_i(N, \bar{v}) \]
\[ = \bar{v}(N) \]
\[ = v(\bar{N}) \]

where the first equality follows from Eq. (1) and the second from the efficiency of the Shapley value. \( \square \)

The next lemma will be used to prove the consistency (Axiom 4) of restricted Shapley values when \( \Omega \) is a lattice.

**Lemma 3.** Let \( \Omega \) be a lattice. Then, for coalitions \( N \) and \( U \subseteq N \) with \( N \setminus U \in \Omega \), we have \( \bar{v}_{\phi^\Omega}(N, U) = v_{\phi^\Omega}(N, U) \).

In the proof of this lemma, we need the following property of a lattice of effective coalitions.

**Lemma 4.** Let \( \Omega \) be a lattice. Let \( N \) be a coalition, \( U \subseteq N \) with \( N \setminus U \in \Omega \). Let \( S \subseteq N \setminus U \). Then, we have the following:

(a) \( S \cup U \subseteq \bar{S} \cup U \);

(b) 0-players of \( S \) are 0-players of \( S \cup U \).

**Proof.** Using the intersection-closedness of \( \Omega \) and \( N \setminus U, \bar{S} \cup \bar{U} \in \Omega \), we obtain \( \bar{S} \cup \bar{U} \cap N \setminus U \in \Omega \). Therefore, we have

\[ \bar{S} \cup \bar{U} = (\bar{S} \cup \bar{U} \cap N \setminus U) \cup (\bar{S} \cup \bar{U} \cap U) \]
\[ \subseteq \bar{S} \cup U \]

This proves statement (a).

Statement (b) follows from

\[ S \setminus \bar{S} \subseteq S \setminus (\bar{S} \cup U) \subseteq S \bar{S} \cup \bar{U} \subseteq (S \cup U)(\bar{S} \cup U) \]

which concludes the proof. \( \square \)
Proof of Lemma 3. From the definition of $\varphi^\Omega$, it is evident that
\[ \varphi^\Omega_i(S, v) = \varphi^\Omega_i(T, v) = \varphi^\Omega_i(S, v) \tag{2} \]
for all coalitions $S$, $T$, with $S \subseteq T \subseteq S$, and players $i \in S$. Now, let $S \subseteq N \setminus U$ be arbitrary. Then, we have
\[
\overline{v}_{\varphi^\Omega}(S) = v_{\varphi^\Omega}(S)
\]
\[ = \sum_{i \in S} \varphi^\Omega_i(\overline{S} \cup U, v) \]
\[ = \sum_{i \in S} \varphi^\Omega_i(S \cup U, v) \]
\[ = \sum_{i \in S} \varphi^\Omega_i(S \cup U, v) \]
\[ = v_{\varphi^\Omega}(S) \]
where the last and first two equalities follow by definition, the third equality follows from Lemma 4(a) and Eq. (2), and the fourth equality follows from Lemma 4(b) and the observation that 0-players are assigned 0 by $\varphi^\Omega$. 

4. The characterization result

In this section, we prove the main result of the paper, yielding the class of all values that satisfy Axioms 1–5. Independence of the axioms follows from examples collected in Appendix A.

Theorem 1. A value $\psi$ on $\mathcal{G}$ satisfies Axioms 1–5 if, and only if $\psi = \varphi^\Omega$ for a lattice $\Omega$.

We first show that every $\varphi^\Omega$ with $\Omega$ a lattice satisfies the five axioms.

Proposition 1. Let $\Omega$ be a lattice. Then, $\varphi^\Omega$ satisfies Axioms 1–5.

Proof. Let $S$ be a non-empty coalition. If $S \not\subseteq \Omega$, then $S$ is $\varphi^\Omega$-ineffective, by definition of $\varphi^\Omega$. If $S \subseteq \Omega$, then let $i \in S$ and let $(N, v)$ be a game with $S \subseteq N$. By Lemma 1, we have
\[ \varphi^\Omega_i(N, v) = \sum_{T: T \subseteq N, i \not\in T} \frac{|T|!(|N| - |T| - 1)!}{|N|!} [v(T \cup \{i\}) - v(T)] \]
Observe that the numbers $v(T)$ never change if we vary $v(S)$, since $i \in S$ but $i \not\in \overline{T}$. However, $v(T \cup \{i\})$ changes if we vary $v(S)$ when $T \cup \{i\} = S$ and, in particular, for $T = S\cup i$. Therefore, $\varphi^\Omega_i(N, v)$ changes when $v(S)$ is varied. We
conclude that $S$ is $\varphi^\Omega$-effective. Thus, $\varphi^\Omega$ satisfies Axiom 1. Also, since $\Omega$ is the collection of all $\varphi^\Omega$-effective coalitions, Axiom 2 is satisfied, since $\Omega$ is a lattice, by assumption. Axiom 3 follows by Lemma 2. Axiom 5 follows by the definition of $\varphi^\Omega$, as is easily established (alternatively, Lemma 1 may be applied).

To show that $\varphi^\Omega$ satisfies Axiom 4, let $(N, v) \in \mathcal{G}$, $U \subseteq N$, $N \setminus U \in \Omega$. Observe that $v_{\varphi^\Omega, U} = \tilde{v}_{\varphi, U}$, since, for arbitrary coalitions $S \subseteq N \setminus U$, we have

$$v_{\varphi^\Omega, U}(S) = \sum_{i \in S} \varphi^\Omega_i(S \cup U, v)$$

$$= \sum_{i \in S} \varphi_i(S \cup U, \tilde{v})$$

$$= \tilde{v}_{\varphi, U}(S)$$

Therefore, we have

$$\varphi^\Omega_i(N \setminus U, v_{\varphi^\Omega, U}) = \varphi_i(N \setminus U, v_{\varphi^\Omega, U})$$

$$= \varphi_i(N \setminus U, \tilde{v}_{\varphi, U})$$

$$= \varphi_i(N, \tilde{v})$$

$$= \varphi^\Omega_i(N, v)$$

where the last and the first equalities are by definition; the second equality is by Lemma 3, the third is by Eq. (3), and the fourth equality follows from the consistency of the Shapley value (Hart and Mas-Colell, 1989). We conclude that $\varphi^\Omega$ satisfies Axiom 4. 

Observe that intersection-closedness of $\Omega$ is only used to prove that $\varphi^\Omega$ satisfies Axiom 4. Indeed, if $\Omega$ is only a power structure and not a lattice, then $\varphi^\Omega$ does not satisfy this axiom. This follows from Theorem 1 and the fact that, for a power structure $\Omega$, the value $\varphi^\Omega$ satisfies Axioms 1–3 and Axiom 5.

We next prove that Axioms 1–5 determine the class of restricted Shapley values that correspond to lattices.

Until further notice, we make the following assumptions. By $\psi$, we denote a value on $\mathcal{G}$ that satisfies Axioms 1–5. $\Omega$ is the collection of coalitions that are $\psi$-effective. (In)effectivity always refers to the value $\psi$, and every reference to a power structure refers to $\Omega$. In particular, for a coalition $N$, if $\bar{N} \neq \emptyset$, then $\bar{N}$ is the unique maximal effective coalition in $N$ and, if $\bar{N} = \emptyset$, then there are no effective coalitions in $N$.

We will prove that $\psi = \varphi^\Omega$. We start with a lemma concerning 0-players.

**Lemma 5.** Let $(N, v) \in \mathcal{G}$. Then, we have the following:

(a) $\psi_i(N, v) = 0$ for every $i \in \mathcal{N} \setminus \bar{N}$;

(b) $\psi_i(N, v) = \psi_i(\bar{N}, v)$ for every $i \in \bar{N}$.
Proof. Part (a) is simply Axiom 5(a). For part (b), assume that $U := N \setminus \tilde{N} \neq \emptyset$. Observe that, for every $S \subseteq \tilde{N}$, we have $S \cup U = \tilde{S}$. Hence, by Axioms 3 and 5(a), for every such $S$, we have

$$v_{\phi, U}(S) = \sum_{i \in S} \psi_i(S \cup U, v)$$

$$= v(\tilde{S}) - \sum_{i \in U} \psi_i(S \cup U, v)$$

$$= v(\tilde{S})$$

(4)

This implies that $v_{\phi, U}$ and $v$ coincide on the effective coalitions in $\tilde{N}$, so that $\psi(\tilde{N}, v_{\phi, U}) = \psi(\tilde{N}, v)$, according to Axiom 1. By Axiom 4, for all $i \in \tilde{N}$, $\psi_i(N, v) = \psi_i(\tilde{N}, v_{\phi, U})$. Therefore, $\psi_i(N, v) = \psi_i(\tilde{N}, v)$ for every $i \in \tilde{N}$. □

Our next objective is to prove that $\Omega$ is in fact a lattice. To prove this, we need to know the distribution $\psi(N, v)$ on games $(N, v)$ in which all coalitions (except possibly $N$) have worth 0. We call such a game a plain game.

Lemma 6. In a plain game, the value $\psi$ assigns to each player the same amount.

Proof. The proof is by induction on the number of players. The lemma is obviously true for 1-player games. Assume that the lemma is true for plain games with less than $n$ players ($n \geq 2$), and let $(N, v)$ be a game with $|N| = n$.

If $N$ contains 0-players, then, because the game $(\tilde{N}, v)$ is also plain, according to the induction hypothesis, all players in $\tilde{N}$ obtain the same amount under $\psi$. Axiom 3 implies that the sum of these amounts equals $v(\tilde{N}) = 0$. Therefore, $\psi_i(\tilde{N}, v) = 0$ for all $i \in \tilde{N}$. By Lemma 5(a) and (b), we conclude that $\psi_i(N, v) = 0$ for all $i \in N$.

If $N$ possesses 1-players, say $U \neq \emptyset$ is the set of 1-players in $N$, then consider the set $M = N \setminus U$. A player in $M$ but not in $\tilde{M}$ would be a 1-player and, therefore, belong to $U$ – an obvious impossibility. Thus, $M = \tilde{M}$, i.e. $M \in \Omega$. By Axiom 5(b), we have

$$\psi_i(N, v) = \frac{1}{n} [v(N) - v(M)] = \frac{1}{n} v(N)$$

(5)

Therefore, by Axiom 3, we have

$$v_{\phi, U}(M) = \sum_{i \in M} \psi_i(N, v) = v(N) - \sum_{i \in U} \psi_i(N, v) = \frac{|M|}{n} v(N)$$

For $S \subseteq M$, we have $v_{\phi, U}(S) = 0$ by the induction hypothesis and Axiom 3 applied to the plain game $(S \cup U, v)$, which has all coalition worths equal to 0. Thus, $v_{\phi, U}$ is a plain game and, by the induction hypothesis and Axiom...
3, \( \psi_i(M, v_{\psi,u}) = (1/n)v(N) \) for every \( i \in M \). This, together with Axiom 4 and Eq. (5), proves that \( \psi_i(N, v) = \psi_j(N, V) \) for all \( i, j \in N \).

Finally, suppose that \( N \) is effective without 1-players. By the same argument as above, \((M, v_{\psi,N,M})\) is a plain game for every \( M \in \sigma \). Therefore, the induction hypothesis and Axiom 4 imply that

\[
\psi_i(N, v) = \psi_j(N, v), \quad \text{for all } i, j \in M, \text{ for all } M \in \Omega \tag{6}
\]

Since there are no 1-players, we can write \( N \) as a union of effective coalitions unequal to \( N \). Since \( \sigma \) is union-closed, we can find \( S, T \neq N \) in \( \sigma \) with \( N = S \cup T \).

If \( S \cap T \neq \emptyset \), let \( i \in S \cap T \). By Eq. (6), \( \psi_i(N, v) = \psi_j(N, v) \) for all \( j \in S \) and \( j \in T \), so we are finished. If \( S \cap T = \emptyset \) and there is an effective coalition \( M \) in \( N \) with \( M \notin \{S, T, N\} \), then without loss of generality, assume \( M \cap S \neq \emptyset \) and let \( i \in M \cap S \). Again, by Eq. (6), \( \psi_i(N, v) = \psi_j(N, v) \) for all \( j \in S \) and \( j \in M \cup T \), so we are finished in this case as well.

The only case left to consider is the situation of Axiom 5(c). In that case, Axiom 5(c) can be applied: since \((N, v)\) is plain, this axiom implies that \( \psi_i(N, v) = (1/n)v(N) \) for every \( i \in N \). This completes the proof of the lemma.

\[ \square \]

**Proposition 2.** If \( \psi \) satisfies Axioms 1–5, then \( \Omega \) is a lattice.

**Proof.** Suppose that \( \Omega \) is not a lattice. Then, there are effective coalitions \( S, T \) with non-empty intersection but not in \( \Omega \). Let \( S, T \) be such coalitions with \( N := S \cup T \) of minimal cardinality, i.e. within \( N \), there is no smaller effective coalition that is the union of two effective coalitions with non-empty intersection not in \( \Omega \).

We claim that there is a player \( i \) in \( S \cap T \) who is a 1-player in \( S \) and also in \( T \). If not, then, for every \( i \in S \cap T \), there is an \( S_i \subset S \) with \( i \in S_i \) and \( S_i \in \Omega \); hence, \( S_i \cap T \in \Omega \), or a \( T_i \subset T \) with \( i \in T_i \) and \( T_i \in \Omega \); hence, \( T_i \cap S \in \Omega \). (Observe that the minimality of \( N \) is used to conclude \( S_i \cap T \in \Omega \) or \( T_i \cap S \in \Omega \).) Hence, every \( i \in S \cap T \) is a member of an effective coalition contained in \( S \cap T \). However, then, by union-closedness, \( S \cap T \in \Omega \), which is a contradiction. This proves our claim.

Let \( i \) be a player as in the previous paragraph. Let \((N, v)\) be a game with, for all \( W \subset N \), if \( i \notin W \), then \( v(W) = 0 \). Consider the reduced game \((S, v_{\psi,T\cap S})\). For \( W \subset S, W \in \Omega \), we have

\[
v_{\psi,T\cap S}(W) = \sum_{j \in W} \psi_j(W \cup (T \cap S), v) = 0
\]

where the second equality follows from Lemma 6, Axiom 3 and the fact that \((W \cup (T \cap S), v)\) is a zero-game: observe that \( i \notin W \cup (T \cap S) \). Hence, \((S, v_{\psi,T\cap S})\) is a plain game. Again using Lemma 6, Axiom 3 and \( S \in \Omega \), we obtain \( \psi_i(S, v_{\psi,T\cap S}) = v_{\psi,T\cap S}(S)/|S| \) for every \( j \in S \). In particular by Axiom 4, \( \psi_i(N, v) = \psi_j(N, v) \) for all
k, j \in S$. Similarly, one proves that $\psi_k(N, v) = \psi_j(N, v)$ for all $k, j \in T$. Because $S \cap T \neq \emptyset$, it follows that $\psi_k(N, v) = \psi_j(N, v)$ for all $k, j \in N$. Hence, by Axiom 3, $\psi_j(N, v) = v(N)/|N|$ for all $j \in N$. This contradicts the effectivity of $S$ and $T$.

We have proved that the restriction of $\Omega$ to $N$ is a lattice. \hfill \Box

Before we can prove Theorem 1, we need one more lemma.

**Lemma 7.** Let $N \subseteq Z$. Suppose that, for all $M \subseteq N$ and all $(M, v) \in \mathcal{G}$, we have

$$\psi(M, v) = \varphi^\alpha(M, v)$$

(7)

Then, $\psi(N, v) = \varphi^\alpha(N, v)$ for all $(N, v) \in \mathcal{G}$.

**Proof.** If $N$ is not effective, then the desired result follows from Lemma 5 applied to $\psi$ and $\varphi^\alpha$. Therefore, assume that $N$ is effective. Let $M \subseteq N$, $M \in \Omega$. For $S \subseteq M$, we have by Eq. (7) that

$$v_{\varphi,N\setminus M}(S) = \sum_{i \in S} \psi_i(S \cup (N\setminus M), v)
= \sum_{i \in S} \varphi^\alpha_i(S \cup (N\setminus M), v)
= v_{\varphi^\alpha,N\setminus M}(S)$$

Let $u_M$ denote the unanimity game on $M$, i.e. $u_M(S) = 1$ if $S \subseteq M$, and $u_M(S) = 0$ otherwise. Then, we can write

$$v_{\varphi^\alpha,N\setminus M} = v_{\varphi,N\setminus M} + \alpha_M^v u_M$$

where $\alpha_M^v = v_{\varphi^\alpha,N\setminus M}(M) - v_{\varphi,N\setminus M}(M)$. By definition of $\varphi^\alpha$ and Eq. (7), it follows that, for all $i \in M$, we have

$$\varphi^\alpha_i(M, v_{\varphi^\alpha,N\setminus M}) = \varphi^\alpha_i(M, v_{\varphi,N\setminus M}) + \alpha_M^v/|M|
= \psi_i(M, v_{\varphi,N\setminus M}) + \alpha_M^v/|M|$$

Hence, we have

$$\varphi^\alpha_i(N, v) = \psi_i(N, v) + \alpha_M^v/|M|$$

(8)

where we use Axiom 4 for both $\psi$ and $\varphi^\alpha$.

If $N$ contains 1-players, say $U \neq \emptyset$ is the coalition of 1-players of $N$, then $M = N \cup U$ is an effective coalition (see the proof of Lemma 6). The 1-players in $N$ obtain the same amounts from both $\varphi^\alpha$ and $\psi$, according to Axiom 5(b). Thus, by Axiom 3, the total amounts for the players in $M$ from $\varphi^\alpha$ and $\psi$ are also the same, implying with Eq. (8) that $\alpha_M^v = 0$ and also that the players in $M$ receive the same from $\varphi^\alpha$ and $\psi$. This proves the lemma for the case where $N$ contains 1-players.

Now, suppose that there are also no 1-players in $N$. Then, we may write...
\[ N = \bigcup_{i \in N} M_i, \text{ where, for each } i \in N, M_i \text{ is an effective coalition with } i \in M_i \subseteq N. \]

Suppose, without loss of generality, that \( \alpha^{\psi}_{M_i} > 0 \) for some \( i \in N \). By Eq. (8) for \( M = M_i \), this implies that \( \varphi^{\Omega}_{j}(N, v) > \psi_{j}(N, v) \) for all \( j \in M_i \). Hence, for all effective \( M \subseteq N \) with \( M \cup M_i \neq N \), we have (also by Eq. (8))

\[ \varphi^{\Omega}_{j}(N, v) > \psi_{j}(N, v), \quad \text{for all } j \in M \cup M_i \]  

From Eq. (9) and the fact that it is impossible that \( \varphi^{\Omega}(N, v) > \psi(N, v) \) by Axiom 3, there is a player \( j \in N \) with \( M_i \cup M_j = N \) and \( M_i \cap M_j = \emptyset \). Note that there is no \( M \subseteq M_i \) with \( M \in \Omega \); otherwise, we would obtain a contradiction by considering \( M \cup M_i \) and applying Eq. (9). Similarly, there is no \( M \subseteq M_j \) with \( M \in \Omega \). Since, by Proposition 2, \( \Omega \) is a lattice, it follows that \( M_i, M_j \) and \( N \) are the only effective coalitions in \( N \). Therefore, the conditions in Axiom 5(c) are satisfied, implying \( \alpha^{\psi}_{M_i} = 0 \), i.e. a contradiction. This completes the proof. \( \square \)

**Proof of Theorem 1.** That \( \varphi^{\Omega} \) satisfies all axioms if \( \Omega \) is a lattice is Proposition 1. The converse follows from Lemma 7, noting that Eq. (7) holds for coalitions that consist of one player, in view of Axiom 5(a) and (b). \( \square \)

Independence of the axioms is proved in Appendix A.

5. **Concluding remarks**

The present paper is related to Hart and Mas-Colell (1989), which presents a characterization of the Shapley value by means of two axioms: consistency, and standardness. The standardness axiom fixes the value for two-player games; our Axiom 5 reduces to this axiom if all coalitions are effective, as is the case with the Shapley value. Besides, we also use an additional efficiency axiom; Hart and Mas-Colell implicitly assume efficiency, not only in their standardness axiom but also in their definition of the consistency property, since they have \( v_{\psi, U}(S) = v(S \cup U) - \sum_{i \in U} \psi_{i}(S \cup U, v) \) in their definition of the reduced game. Finally, our Axioms 1 and 2 are specific for our restricted coalitions model. As remarked in the Introduction, a new proof of the uniqueness part of the Hart–Mas-Colell characterization of the Shapley value can be derived from the proof of our main result; this new proof is based on the second part of the proof of Lemma 7.

Of the existing characterizations of the Shapley value, some can be extended to restricted Shapley values in quite a direct manner. This is true, for instance, for characterizations that involve linearity or additivity axioms (see Bäcker, 1995). Another characterization can be based on the following axiom of ‘balanced contributions’ (cf. Myerson, 1980; Hart and Mas-Colell, 1989).\(^2\)

\(^2\) This characterization has been suggested to us by a referee of the paper.
Axiom 6. For all \((N, v) \in \mathcal{G}\) and all \(i, j \in N\), we have
\[
\psi_i(N, v) - \psi_i(N \setminus \{j\}, v) = \psi_j(N, v) - \psi_j(N \setminus \{i\}, v)
\]

Instead of the whole Axiom 5, it will suffice to impose only part (a), dealing with 0-players. We then have the following characterization.

Theorem 2. A value \(\psi\) on \(\mathcal{G}\) satisfies Axioms 1–3, 5(a) and 6 if and only if \(\psi = \varphi^\Omega\) for a power structure \(\Omega\).

We omit the proof of this theorem, which is straightforward and can be based on induction. It can also be shown that the axioms are independent. Note that, in this characterization, \(\Omega\) does not have to be a lattice. Actually, one could even relax Axiom 2, i.e. union-closedness of the class of effective coalitions, to obtain a larger class of restricted Shapley values.

Appendix A: independence of the axioms

We present seven examples of values, satisfying all axioms except for Axioms 1, 2, 3, 4, 5(a), 5(b) and 5(c), respectively, i.e. example \(i\) satisfies all axioms except for Axiom \(i\). Proofs are left to the reader.

(1) Let \(Z = \{1, 2\}\), \(\Omega = \{\{2\}, \{1, 2\}\}\), and let \(\psi\) be given by \(\psi(\{1, 2\}, v) = \varphi^\Omega(\{1, 2\}, v)\) for all \((\{1, 2\}, v) \in \mathcal{G}\), \(\psi(1, v) = v(1)\), \(\psi(2, v) = v(2)\) for all \((1, v), (2, v) \in \mathcal{G}\). Coalition \(\{1\}\) is neither effective nor ineffectiv.

(2) Let \(Z = \{1, 2, 3\}\) and, for every \((N, v) \in \mathcal{G}\), let \(\psi(N, v) = \varphi(N, \bar{v})\), where \(\varphi\) is the Shapley value and \(\bar{v}(S) := v(S)\) if \(S \neq Z\), \(\bar{v}(Z) := v(\{1, 2\})\). All coalitions are effective except for \(Z\).

(3) Let \(Z = \{1, 2\}\), \(\psi_1(\{1, 2\}, v) = 0\) and \(\psi_2(\{1, 2\}, v) = (1/2)[v(\{1, 2\}) - v(1)]\) for all \((\{1, 2\}, v) \in \mathcal{G}\), \(\psi(1, v) = v(1)\) for all \((1, v) \in \mathcal{G}\), \(\psi(2, v) = 0\) for all \((2, v) \in \mathcal{G}\). Efficiency for \(\{1, 2\}\) is violated.

(4) Let \(Z = \{1, 2, 3\}\), \(\psi(N, v) = \varphi(N, v)\) for all \((N, v) \in \mathcal{G}\) with \(|N| < 3\) (where \(\varphi\) is again the Shapley value), and
\[
\begin{align*}
\psi_1(\{1, 2, 3\}, v) & = v(1) + v(\{1, 2\}) + v(\{1, 3\}) + \alpha/3 \\
\psi_2(\{1, 2, 3\}, v) & = v(2) + v(\{1, 2\}) + v(\{2, 3\}) + \alpha/3 \\
\psi_3(\{1, 2, 3\}, v) & = v(3) + v(\{1, 3\}) + v(\{2, 3\}) + \alpha/3
\end{align*}
\]
for all \((\{1, 2, 3\}, v) \in \mathcal{G}\), where \(\alpha = v(N) - [v(1) + v(2) + v(3) + 2v(\{1, 2\}) + 2v(\{1, 3\}) + 2v(\{2, 3\})]\). This value \(\psi\) violates Axiom 4.

(5(a)) Let \(Z = \{1, 2, 3\}\), \(\Omega = \{\{1\}, \{2\}, \{1, 2\}\}\) and, for all \((Z, v) \in \mathcal{G}\), let
\[ \psi_1(Z, v) = \frac{\varphi_1^0(Z, v)}{2} \]
\[ \psi_2(Z, v) = \frac{\varphi_2^0(Z, v)}{2} \]
\[ \psi_3(Z, v) = \frac{v(\{1, 2\})}{2} \]

Furthermore, let \( \psi(N, v) = \varphi^0(N, v) \) for all \((N, v) \in \mathcal{G} \), with \(|N| < 3\). This value \( \psi \) violates Axiom 5(a), but not (b) or (c).

(5(b)) Let \( Z = \{1, 2\} \) and, for all \((\{1, 2\}, v) \in \mathcal{G} \), let
\[ \psi_1(\{1, 2\}, v) = \frac{2}{3} v(\{1, 2\}) + \frac{1}{3} v(1) \]
\[ \psi_2(\{1, 2\}, v) = \frac{1}{3} v(\{1, 2\}) - \frac{1}{3} v(1) \]

Let \( \psi(1, v) = v(1) \) and \( \psi(2, v) = 0 \) for all \((1, v), (2, v) \in \mathcal{G} \). This value \( \psi \) violates Axiom 5(b), but not (a) or (c).

(5(c)) Let \( Z = \{1, 2\} \) and, for all \((\{1, 2\}, v) \in \mathcal{G} \), let
\[ \psi_1(\{1, 2\}, v) = \frac{1}{3} v(\{1, 2\}) + \frac{1}{2} [v(1) - v(2)] \]
\[ \psi_2(\{1, 2\}, v) = \frac{2}{3} v(\{1, 2\}) - \frac{1}{2} [v(1) - v(2)] \]

Let \( \psi(1, v) = v(1) \) and \( \psi(2, v) = v(2) \) for all \((1, v), (2, v) \in \mathcal{G} \). This value \( \psi \) violates Axiom 5(c), but not (a) or (b).

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