Generalized Median Solutions, Strategy-Proofness and Strictly Convex Norms

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Abstract: Generalized location problems with $n$ agents are considered, who each report a point in $m$-dimensional Euclidean space. A solution assigns a compromise point to these $n$ points, and the individual utilities for this compromise point are equal to the negatives of the distances to the individual positions. These distances are measured by a given strictly convex norm, common to all agents. For $m = 2$, it is shown that if a Pareto optimal, strategy-proof and anonymous solution exists, then $n$ must be odd, and the solution is obtained by taking the median coordinate-wise, where the coordinates refer to a basis that is orthogonal with respect to the given norm. Furthermore, in that case ($m = 2$) such a solution always exists. For $m > 2$, existence of a solution depends on the norm.

Key Words: Strategy-proofness, median solutions

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(ideal point, bliss point). These distances are determined by a given strictly convex norm; the same norm is assumed for all players. We will focus on solutions that are anonymous, and Pareto optimal and strategy-proof with respect to the given norm or, equivalently, distance function. The first property means that the solution is invariant under permutations of the agents, i.e., the names of the agents do not matter. Pareto optimality means that for any given profile (n-type of reported points) there is no point that is at least as good as the compromise point for all agents and strictly preferred by at least one agent. Strategy-proofness means that no agent is better off by being dishonest, i.e., not reporting his true bliss point, whatever the other agents report.

For the case $m = 1$, the class of all solutions with these three properties has been characterized by Moulin (1980). He shows that any such solution must be of the following form. Choose $n - 1$ points $a_1, a_2, \ldots, a_{n-1}$ in $\mathbb{R} \cup \{-\infty\} \cup \{\infty\}$. If the agents report the points $x_1, x_2, \ldots, x_n$, then the solution assigns the median point of the $2n - 1$ points $a_1, \ldots, a_{n-1}, x_1, \ldots, x_n$.

For the Euclidean norm, the case $m > 1$ was investigated in PSS² (1992). It was proved that solutions with the mentioned three properties only exist if $m = 2$ and $n$ (the number of agents) is odd. In this case, any anonymous, Pareto optimal and strategy-proof solution must be of the following form. Choose an orthogonal basis (an orthogonal pair of axes) in the plane. Coordinates of points are chosen with respect to this basis (i.e., projections on these axes). Of the reported points $x^1, \ldots, x^n$, determine the two median coordinates; these coordinates are the coordinates of the solution point. Kim and Roush (1984) consider continuity instead of Pareto optimality; see section 6 for further discussion.

In the present paper these results are extended to the case where the distance function is derived from any strictly convex norm. Strict convexity means that the unit disc is a strictly convex set. The results are to some extent analogous to those in case of the Euclidean norm, but the extension is far from being straightforward. One of the difficulties lies in the fact that orthogonality is only defined for norms derived from an inner product, i.e., the Euclidean norm and the norms derived from the Euclidean norm by a nonsingular linear transformation.

Briefly, the results are as follows. Again, if $m = 2$, an anonymous, Pareto optimal and strategy-proof solution exists only if $n$ is odd. In that case, it must assign coordinate-wise medians with respect to a basis that is orthogonal (in a way to be defined in the next section) with respect to the given norm.

In the paper, most attention will be paid to the 2-dimensional case ($m = 2$). A proof of the impossibility result for $m > 2$ and the Euclidean norm was given in PSS (1992, Theorem 4.2). For general strictly convex norms, this impossibility result no longer holds; see section 6.

In more general terms, a location problem is a social choice problem with an infinite number of alternatives, even a continuum, but a restricted domain of admissible preferences. In that context, a solution is called a social choice

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² Peters, H., H. van der Stel, and T. Storekken.
function. Social choice theory has been plagued by so-called impossibility results, starting with Arrow (1963), who showed that, if there are at least two agents and three alternatives and if the set of admissible preferences consists of all linear orders on the (finite) set of alternatives, then any Pareto optimal social choice rule (assigning a social preference rather than an alternative) satisfying Arrow’s Independence of Irrelevant Alternatives condition must be dictatorial. Later, Gibbard (1973) and Satterthwaite (1975) independently showed that, in the same framework, any surjective strategy-proof social choice function must be dictatorial. Since then, many authors have tried to obtain possibility results, and the present paper belongs to this stream. Further discussion of related literature is postponed until section 7.

The organization of this paper is as follows. Section 2 contains a formal description of the model and of the main results. Readers not interested in the proofs of these results may restrict their attention to this section. Section 3 deals with the case where profiles are restricted to two locations. Section 4 gives the proof of the case with three locations. The extension of this to the general case is much less work, performed in Section 5. In Section 6 it is shown that also for \( m > 2 \) anonymous, Pareto optimal and strategy-proof solutions may exist. Section 7 concludes with further discussion of related literature.

2 Model and Results

2.1 The Model

Let \( N = \{1, 2, \ldots, n\} \), with \( n \geq 1 \), denote the set of agents. An element \( p \) of \((\mathbb{R}^2)^N\) is called a profile. The component \( p(i) \) of \( p \) will be called the preference vector of agent \( i \). A solution is a map \( \varphi: (\mathbb{R}^2)^N \to \mathbb{R}^2 \). Given such a map, \( \varphi(p) \) is called the solution point for \( p \).

Let \( \delta \) be a metric on \( \mathbb{R}^2 \) induced by a strictly convex norm \( \| \cdot \| \). A norm \( \| \cdot \| \) is called strictly convex if for all \( a, b, c \in \mathbb{R}^2 \) such that \( c \notin \text{conv}\{a, b\} \), we have \( \|a - c\| + \|b - c\| > \|a - b\| \). This definition is equivalent to the standard definition in functional analysis, see for instance Kreyszig (1978, p. 332). We will call also \( \delta \) strictly convex.

We are interested in solutions \( \varphi \) with the properties described below:

- **Anonymity** (AN): \( \varphi(p \circ \sigma) = \varphi(p) \) for every \( p \in (\mathbb{R}^2)^N \) and every permutation \( \sigma: N \to N \).

- **Pareto Optimality (with respect to \( \delta \))** (PO): For no \( p \in (\mathbb{R}^2)^N \) there is an \( x \in \mathbb{R}^n \) with \( \delta(x, p(i)) \leq \delta(\varphi(p), p(i)) \) for all \( i \in N \) such that at least one of these inequalities is strict.

\(^2\) "conv" denotes "the convex hull of".
Strategy-proofness (with respect to $\delta$) (SP): For all $i \in N$ and all profiles $p$, $\beta \in (\mathbb{R}^2)^N$ with $p(j) = \beta(j)$ for all $j \neq i$, we have:

$$\delta(p(i), \varphi(p)) \leq \delta(p(i), \varphi(\beta)).$$

Intermediate Strategy-proofness (with respect to $\delta$) (ISP): For all $M \subset N$, $x \in \mathbb{R}^2$, and $p, \beta \in (\mathbb{R}^m)^N$ with $p(M) = \{x\}$ and $p(j) = \beta(j)$ for all $j \in N - M$, we have:

$$\delta(x, \varphi(p)) \leq \delta(x, \varphi(\beta)).$$

Anonymity requires that interchanging agents does not influence the solution point. Pareto optimality means that no agent can gain without some other agent loosing, and strategy-proofness states it is always optimal to be honest about one's preferences. Intermediate strategy-proofness states that if all agents belonging to some subset $M \subset N$ have the same preference vector, they do not gain by choosing different preference vectors.

The properties Pareto optimality and strategy-proofness can be characterized in a more convenient way, as will be shown by the next two lemmas. We will use these characterizations throughout the paper, often without explicit mentioning.

Lemma 2.1: Let $\varphi$ be a solution. Then the following assertions are equivalent:

(i) $\varphi$ satisfies PO.
(ii) $\varphi(p) \in \text{conv}\{p(1), \ldots, p(n)\}$ for all $p \in (\mathbb{R}^2)^N$.

The proof of this lemma is given in PSS (1993).

Lemma 2.2: Let $\varphi$ be a solution. Then the following assertions are equivalent:

(i) $\varphi$ satisfies SP
(ii) $\varphi$ satisfies ISP.

Proof: The implication (ii) $\Rightarrow$ (i) is evident. For the converse implication, let $\varphi$ be strategy-proof with respect to $\delta$, $M \subset N$, $x \in \mathbb{R}^m$ and $p, \beta \in (\mathbb{R}^m)^N$, such that $p(M) = \{x\}$ and $p(j) = \beta(j)$ for all $j \in N - M$. We will prove that $\delta(x, \varphi(p)) \leq \delta(x, \varphi(\beta))$. Without loss of generality suppose $M = \{1, 2, \ldots, l\}$. Take $p^0$ up to $p^l$ defined for all $i \in \{1, \ldots, l\}$ and $j \in N$ as follows:

$$p^l(j) = \begin{cases} p(j) & \text{if } j > i \\ \beta(j) & \text{if } j \leq i. \end{cases}$$

Then it follows from the strategy-proofness of $\varphi$ that for all $i \in M$: $\delta(p^{l-1}(i), \varphi(p^{l-1})) \leq \delta(p^{l-1}(i), \varphi(p^l)).$
Hence by the definition of $p^0$ up to $p'$ it follows that for all $i \in S$: $\delta(x, \varphi(p_i^{i-1})) \leq \delta(x, \varphi(p_i))$, hence, by the transitivity of $\leq$:

$$\delta(x, \varphi(p)) = \delta(x, \varphi(p^0)) \leq \delta(x, \varphi(p')) = \delta(x, \varphi(\hat{p})) .$$

We introduce the following notations, for $x, y, z \in \mathbb{R}^2$:

- $[x, y] := \text{conv}(x, y)$
- $[x, y] := [x, y] \setminus \{y\}$
- $(x, y) := [x, y] \setminus \{x, y\}$
- $[x, y, \rightarrow) := \{u \in \mathbb{R}^2 | \exists \lambda \in [0, \infty): u = x + \lambda(y - x)\}$
- $(x, y, \rightarrow) := [x, y, \rightarrow) \setminus \{x\}$
- $(\leftarrow, x, y, \rightarrow) := [x, y, \rightarrow) \cup [y, x, \rightarrow)$
- $\angle(x, y, z) := \text{conv}(x, y, z)$

$\angle(x, y, z)$ denotes the (measure of the) angle between $z - y$ and $x - y$ for $x, y, z$ non-collinear.

We are now able to state an important consequence of strategy-proofness, which will be used throughout the paper.

**Lemma 2.3**: Let $\varphi$ be a solution satisfying SP. Let $p, \hat{p} \in (\mathbb{R}^2)^N$, such that $\hat{p}(i) \in [p(i), \varphi(p)]$ for all $i \in N$. Then $\varphi(\hat{p}) = \varphi(p)$.

**Proof**: It is sufficient to prove the lemma for profiles $p, \hat{p}$ with $\hat{p}(i) \in [\varphi(p), p(i)]$ and $\hat{p}(j) = p(j)$ for all $j \neq i$. For such profiles, by SP:

$$\delta(p(i), \varphi(p)) \leq \delta(p(i), \varphi(\hat{p})) \quad (2.1)$$

and

$$\delta(\hat{p}(i), \varphi(\hat{p})) \leq \delta(\hat{p}(i), \varphi(p)) . \quad (2.2)$$

If $\hat{p}(i) \notin \text{conv}\{p(i), \varphi(\hat{p})\}$ then by strict convexity of $\delta$,
\[ \delta(p(i), \varphi(\beta)) < \delta(p(i), \beta(i)) + \delta(\beta(i), \varphi(\beta)) \].

Hence by (2.2),

\[ \delta(p(i), \varphi(\beta)) < \delta(p(i), \beta(i)) + \delta(\beta(i), \varphi(p)) = \delta(p(i), \varphi(p)). \]

This contradicts (2.1), hence \( \beta(i) \in \text{conv} \{ p(i), \varphi(\beta) \} \). Consequently, (2.1) and (2.2) imply \( \varphi(\beta) = \varphi(p) \). \( \square \)

So strategy-proofness implies that moving preference vectors \( p(i) \) in the direction of \( \varphi(p) \) does not influence the solution point. If, in particular, we replace some \( p(i) \) by \( \varphi(p) \), the solution point of the new profile will be the same. This is an argument which will frequently be used throughout the paper. The property described in lemma 2.3 can be seen as a relaxation of strategy-proofness.

It will often be convenient to use the notation \( \varphi(p(1), \ldots, p(n)) \) instead of \( \varphi(p) \), where all \( p(i) \in \mathbb{R}^2 \). Furthermore, we will use notations like \( \varphi(a^k, b^l) \), where \( k, l \in \mathbb{N} \) are such that \( k + l = n \) or \( \varphi(a^k, b^l, c^m) \), where \( k, l, m \in \mathbb{N} \) are such that \( k + l + m = n \). In the latter case \( \varphi(a^k, b^l, c^m) \) stands for \( \varphi(p) \), where:

\[
\begin{align*}
p(i) &= a & i &= 1, \ldots, k \\
p(i) &= b & i &= k + 1, \ldots, k + l \\
p(i) &= c & i &= k + l + 1, \ldots, n.
\end{align*}
\]

Notice that, if \( \varphi \) is anonymous, it does not matter which preference vectors \( p(i) \) are equal to \( a, b, \) or \( c \); only the number of \( p(i) \) equal to \( a, b, \) or \( c \) matters.

The properties PO, SP and ISP are defined with respect to a metric \( \delta \), where \( \delta \) is induced by a strictly convex norm.

The following lemma characterizes strict convexity of a norm.

**Lemma 2.4:** Let \( \| \cdot \| \) be a norm on \( \mathbb{R}^2 \). Then (i) and (ii) are equivalent, where:

(i) \( \| \cdot \| \) is strictly convex.

(ii) For all \( x, y, z \in \mathbb{R}^2 \) such that \( \| x \| \leq \| y \| \) and \( z \in \{ x, y \} \) we have \( \| z \| < \| y \| \).

The proof of this lemma is given in PSS (1993). In general a strictly convex metric \( \delta \) will not be induced by an inner product. Therefore, we need a generalization of the concept of 'orthogonality' to arbitrary norms. Let \( a, b \in \mathbb{R}^2 \). Then \( a \) and \( b \) are \( \| \cdot \| \)-orthogonal if \( \| a \| \leq \| a + \lambda b \| \) and
\[ \|b\| \leq \|b + \lambda a\| \text{ for all } \lambda \in \mathbb{R}. \] This definition is equivalent to requiring that \( a \) and \( b \) are orthogonal with respect to each other in the sense of Birkhoff (1935). It is easy to verify that in the case where \( \delta \) is induced by an inner product, this definition is consistent with the usual definition of orthogonality.

Given a vector \( a \) there need not be a nonzero vector \( b \) which is \( \| \cdot \| \)-orthogonal to \( a \). However, the next lemma shows that a pair of nonzero \( \| \cdot \| \)-orthogonal vectors always exists.

\textbf{Lemma 2.5:} Let \( \| \cdot \| \) be a norm on \( \mathbb{R}^2 \). Then there are nonzero \( a, b \in \mathbb{R}^2 \) such that \( a \) and \( b \) are \( \| \cdot \| \)-orthogonal.

\textbf{Proof:} Let \( X = \{ x \in \mathbb{R}^2 | \| x \| = 1 \} \). Let \( A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). By the compactness of \( X \) we can take \( a, b \in X \) such that

\[ a^T Ab = \sup_{x \in X, y \in X} x^T Ay. \]

Consequently:

\[ a^T Ab \geq a^T Ax, \quad \forall x \in X. \quad (2.3) \]

By \( a \in X \) we have \( -a \in X \). Therefore \( b^T A(-a) = -b^T Aa = -a^T Ab = \sup_{x \in X, y \in X} x^T Ay \). Hence:

\[ b^T A(-a) \geq b^T Ax, \quad \forall x \in X. \quad (2.4) \]

Suppose for some \( \lambda \in \mathbb{R} \), \( \| a \| > \| a + \lambda b \| \). Hence for some \( \mu \in (0, \infty) \) we have \( \| a + \lambda b + \mu a \| = \| a \| = 1 \). So \( a + \lambda b + \mu a \in X \). Hence, \( -a - \lambda b - \mu a \in X \). On the other hand, \( b^T A(-a) \geq b^T A(\tau A^T b) > 0 \) for \( \tau \in (0, \infty) \) such that \( \| \tau A^T b \| = 1 \). Hence,

\[ b^T A(-a - \lambda b - \mu a) = b^T A(-a) - \lambda b^T A b + \mu b^T A(-a) > b^T A(-a). \]

This contradicts (2.4). Consequently, \( \| a \| \leq \| a + \lambda b \| \) for all \( \lambda \in \mathbb{R} \). Similarly, (2.3) leads to \( \| b \| \leq \| b + \lambda a \| \) for all \( \lambda \in \mathbb{R} \). Therefore, \( a \) and \( b \) are \( \| \cdot \| \)-orthogonal. Furthermore, \( a \) and \( b \) are nonzero because \( a, b \in X \).

The strict convexity of a norm \( \| \cdot \| \) has important implications for the \( \| \cdot \| \)-orthogonality of nonzero vectors.
Lemma 2.6: Let \( \| \cdot \| \) be a strictly convex norm. Let \( a, b \in \mathbb{R}^2 \setminus \{0\} \) be \( \| \cdot \| \)-orthogonal. Then \( \| a \| < \| a + \lambda b \| \) and \( \| b \| < \| b + \lambda a \| \) for all \( \lambda \in \mathbb{R} \setminus \{0\} \).

Proof: Suppose \( \lambda \in \mathbb{R} \setminus \{0\} \) is such that \( \| a + \lambda b \| \leq \| a \| \). Then by lemma 2.4 \( \| a + \frac{\lambda}{2} b \| < \| a \| \). This contradicts the \( \| \cdot \| \)-orthogonality of \( a \) and \( b \). Hence, \( \| a \| < \| a + \lambda b \| \) for all \( \lambda \in \mathbb{R} \setminus \{0\} \). Similarly we can prove: \( \| b \| < \| b + \lambda a \| \) for all \( \lambda \in \mathbb{R} \setminus \{0\} \).

Let \( k \in \{0, 1, 2, \ldots\} \) and let \( x_1, x_2, \ldots, x_{2k+1} \) be real numbers. The median of these numbers, denoted \( \text{med}(x_1, \ldots, x_{2k+1}) \) is the real number \( x \) with \( \#\{i : x_i \leq x\} \geq k + 1 \) and \( \#\{i : x_i \geq x\} \geq k + 1 \).

Let \( a, b \in \mathbb{R}^2 \) be linearly independent. Let \( \lambda, \mu : \mathbb{R}^2 \to \mathbb{R} \) be such that for all \( x \in \mathbb{R}^2 \): \( x = \lambda(x)a + \mu(x)b \). In other words, \( \lambda(x) \) and \( \mu(x) \) are the coordinates of \( x \) with respect to the basis \( \{a, b\} \) of \( \mathbb{R}^2 \). We define the generalized median solution \( \varphi \) with respect to \( \{a, b\} \) as follows:

\[
\varphi(p) = \text{med}(\lambda(p(1)), \ldots, \lambda(p(n))) \cdot a + \text{med}(\mu(p(1)), \ldots, \mu(p(n))) \cdot b.
\]

These generalized median solutions will play a central role in the paper. The following lemma establishes some important properties of these solutions.

Lemma 2.7: Let the number of agents \( n \) be odd. Let \( a, b \in \mathbb{R}^2 \) be linearly independent. Let \( \varphi : (\mathbb{R}^2)^n \to \mathbb{R}^2 \) be the generalized median solution with respect to \( \{a, b\} \). Then:

(i) \( \varphi \) satisfies anonymity and Pareto optimality.
(ii) \( \varphi \) satisfies strategy-proofness if and only if \( a \) and \( b \) are \( \| \cdot \| \)-orthogonal.

Proof: (i) It is clear that \( \varphi \) satisfies anonymity. Let \( \lambda(x) \) and \( \mu(x) \) be the coordinates of \( x \) with respect to the basis \( \{a, b\} \). In other words, \( \lambda, \mu : \mathbb{R}^2 \to \mathbb{R} \) are such that \( x = \lambda(x) \cdot a + \mu(x) \cdot b \) for all \( x \in \mathbb{R}^2 \). Let \( i, j \in N \) with \( \lambda(p(i)) = \lambda(p(j)) \) and \( \mu(p(j)) = \mu(p(j)) \). W.l.o.g.\(^4\) suppose \( \mu(p(i)) < \mu(p(j)) \) and \( \lambda(p(j)) < \lambda(p(j)) \). In order to prove Pareto optimality of \( \varphi \), it is, in view of lemma 2.1, sufficient to show that there must be an \( l \in N \) with \( \lambda(p(l)) \geq \lambda(p(j)) \) and \( \mu(p(l)) \geq \mu(p(j)) \).

We will argue from contradiction: suppose such an \( l \) does not exist. Since there must be at least \( \frac{n + 1}{2} \) agents \( l \in N \) with \( \mu(p(l)) \geq \mu(p(j)) \), we have:

\(^4\) Without loss of generality.
\begin{align*}
\# \{ i \in N | \mu(p(i)) \geq \mu(\phi(p)), \lambda(p(i)) < \lambda(\phi(p)) \} & \geq \frac{n+1}{2}.
\end{align*}

Similarly:

\begin{align*}
\# \{ i \in N | \lambda(p(i)) \geq \lambda(\phi(p)), \mu(p(i)) < \mu(\phi(p)) \} & \geq \frac{n+1}{2}.
\end{align*}

Since these sets are disjoint, we conclude \( n \geq n + 1 \), which is impossible. This establishes the desired contradiction. Thus, \( \phi \) is Pareto optimal.

(ii) Let \( \lambda, \mu : \mathbb{R}^2 \to \mathbb{R} \) be such that \( x = \lambda(x) \cdot a + \mu(x) \cdot b \) for all \( x \in \mathbb{R}^2 \). We first will show that \( \phi \) is strategy-proof if \( a \) and \( b \) are \( \| \cdot \| \)-orthogonal. Let \( i \in N \), and let \( p, \phi \) be profiles with \( p(j) = \phi(j) \) for all \( j \neq i \). Let:

\begin{align*}
\alpha &= \text{med}(\lambda(p(1)), \ldots, \lambda(p(n))) \\
\dot{\alpha} &= \text{med}(\lambda(\phi(1)), \ldots, \lambda(\phi(n))) \\
\beta &= \text{med}(\mu(p(1)), \ldots, \mu(p(n))) \\
\dot{\beta} &= \text{med}(\mu(\phi(1)), \ldots, \mu(\phi(n))).
\end{align*}

Then by definition of the median, \( \dot{\alpha} \leq \alpha \leq \lambda(p(i)) \) or \( \lambda(p(i)) \leq \alpha \leq \dot{\alpha} \), and \( \dot{\beta} \leq \beta \leq \mu(p(i)) \) or \( \mu(p(i)) \leq \beta \leq \dot{\beta} \). \( a \) and \( b \) are \( \| \cdot \| \)-orthogonal, so by lemma 2.4:

\begin{align*}
\| p(i) - \phi(p) \| &= \| p(i) - (\alpha \cdot a + \beta \cdot b) \| \\
&= \| (\lambda(p(i)) - \alpha) \cdot a + (\mu(p(i)) - \beta) \cdot b \| \\
&\leq \| (\lambda(p(i)) - \dot{\alpha}) \cdot a + (\mu(p(i)) - \dot{\beta}) \cdot b \| \\
&\leq \| (\lambda(p(i)) - \dot{\alpha}) \cdot a + (\mu(p(i)) - \dot{\beta}) \cdot b \| \\
&= \| p(i) - (\dot{\alpha} \cdot a + \dot{\beta} \cdot b) \| \\
&= \| p(i) - \phi(\dot{p}) \|.
\end{align*}

Consequently, \( \phi \) is strategy-proof.

Conversely, let \( a \) and \( b \) be not \( \| \cdot \| \)-orthogonal. W.l.o.g. \( \tau \in \mathbb{R} \) is such that \( \| b \| > \| b + \tau a \| \). Let \( p(i) = b - \tau a \) for \( i = 1, \ldots, \frac{n-1}{2} \), \( p(i) = b + \tau a \) for \( i = \)
\[ \frac{n + 1}{2}, \ldots, n - 1, p(n) = 0. \text{ Let } \beta(i) = p(i) \text{ for } i = 1, \ldots, n - 1, \beta(n) = b + \tau a. \]

Then \( \varphi(p) = b, \varphi(\beta) = b + \tau a. \) Hence, \( \|p(n) - \varphi(\beta)\| < \|p(n) - \varphi(p)\|. \) Hence, the generalized median solution with respect to \( \{a, b\} \) is not strategy-proof. \( \square \)

2.2 Main Results

In this section we list the results of this paper.

**Theorem 2.1:** Let \( \delta \) be induced by a strictly convex norm \( \| \cdot \|. \) Let the number of agents \( n \) be odd. Let \( \varphi : (\mathbb{R}^2)^n \to \mathbb{R}^2 \) be a solution. Then \( \varphi \) satisfies anonymity, Pareto optimality and strategy-proofness if and only if \( \varphi \) is the generalized median solution with respect to some linearly independent \( \| \cdot \| \)-orthogonal pair of vectors.

The proof of this theorem will be postponed until section 5. The oddness of the number of agents is essential, as is shown by the next theorem.

**Theorem 2.2:** Let \( \delta \) be induced by a strictly convex norm \( \| \cdot \|. \) Let the number of agents \( n \) be even. Then there is no solution \( \varphi : (\mathbb{R}^2)^n \to \mathbb{R}^2 \) satisfying anonymity, Pareto optimality and strategy-proofness.

The proof of this theorem will be given at the end of section 3.

**Theorem 2.3:** Let \( \delta \) be induced by a strictly convex norm \( \| \cdot \|. \) Let the number of agents \( n \) be odd. Then there exists a solution \( \varphi : (\mathbb{R}^2)^n \to \mathbb{R}^2 \) which satisfies anonymity, Pareto optimality and strategy-proofness.

**Proof:** By lemma 2.5 we can take nonzero \( a, b \in \mathbb{R}^2 \) such that \( a \) and \( b \) are \( \| \cdot \| \)-orthogonal. Let \( \varphi \) be the generalized median solution with respect to \( \{a, b\} \). Then theorem 2.1 implies that \( \varphi \) satisfies AN, PO, and SP. \( \square \)

2.3 Outline of the Proofs

Sections 3, 4 and 5 are devoted to the proof of theorems 2.1 and 2.2. Since these proofs are rather lengthy, their structure will be outlined here.

The property anonymity of \( \varphi \) will often be used, usually without explicit mentioning. Whenever we use Pareto-optimality or strategy-proofness, we will
mention this. However in these cases we frequently use the lemmas 2.1 and 2.2. These lemmas will usually be used without mentioning.

In section 3 we investigate the case where all preference vectors \( p(i) \) are concentrated in two locations. Lemmas 3.1, 3.2 and 3.3 are only used to prove lemma 3.4, which states that in the two location case, the solution point \( \varphi(p) \) is the point where the majority of the \( p(i) \) is concentrated. It is then easy to prove theorem 2.2, because when the number of agents \( n \) is even, there need not be such a point.

In section 4 we study the case where all preference vectors \( p(i) \) are concentrated in three locations. Lemma 4.1 states that if the majority of the \( p(i) \) coincides, the solution point \( \varphi(p) \) will be this point. In the remainder of this section we will assume this is not the case. Hence, we will study \( \varphi(x^k, y^l, z^m) \), where \( x, y, z \in \mathbb{R}^2 \) and \( k, l, m \leq \frac{n - 1}{2} \).

Lemma 4.2 asserts that if all preference vectors are collinear, the solution point \( \varphi(p) \) is the middle point. Lemmas 4.3–4.5 give some basic results that we will use throughout the proof. We will not describe them here.

Lemmas 4.6–4.8 deal with the continuity of \( \varphi \). The main result is given by lemma 4.8, which states that when we take \( k, l, m \) to be constants, \( \varphi \) can be seen as a continuous function of \( x, y \) and \( z \).

Lemmas 4.9 and 4.10 are used to prove lemma 4.11. This lemma is a generalization of lemma 2.3 for the three location case. It asserts that moving \( x, y \) or \( z \) in the direction of \( \varphi(p) \) or in opposite direction does not influence the solution point.

Lemma 4.12 describes a situation which determines the nonzero \( \| \cdot \| \)-orthogonal directions with respect to which \( \varphi \) will turn out to be the generalized median solution. Lemmas 4.13–4.23 are mainly devoted to the proof that a situation as in lemma 4.12 exists. In lemma 4.24 the existence of the desired configuration is proven.

The proof that \( \varphi \) has to be a generalized median with respect to these \( \| \cdot \| \)-orthogonal directions, is now straightforward. The main result for three locations is given in lemma 4.26. Theorem 4.1 summarizes the results for all three location cases. In this lemma \( k, l \) and \( m \) are no longer taken to be fixed.

Finally, theorem 2.1 is proven in section 5.

3 Two Locations

Throughout this section, \( \varphi : (\mathbb{R}^2)^N \to \mathbb{R}^2 \) is a solution satisfying PO, SP and AN with respect to a metric \( \delta \) derived from a strictly convex norm \( \| \cdot \| \), and \( k, l \in \mathbb{N} \) satisfy \( k + l = n \). We will investigate the case where all preference vectors \( p(i) \) are concentrated in two locations. In other words, we will study \( \varphi(u^k, v^l) \), where \( u, v \in \mathbb{R}^2 \).
Lemma 3.1: Let \( u, v \in \mathbb{R}^2 \) be such that \( \varphi(w^k, v') = v \). Then \( \varphi(w^k, v') = v \) for all \( w \in [v, u, \rightarrow) \).

Proof: W.l.o.g. \( u \neq v \). If \( w \in [v, u] \), lemma 2.3 implies \( \varphi(w^k, v') = v \). Now let \( w \notin [v, u] \). First we will assume \( \delta(w, u) < \delta(u, v) \). Pareto optimality implies \( \varphi(w^k, v') \in [w, v] \). By SP, \( \delta(u, \varphi(w^k, v')) \geq \delta(u, v) \), hence \( \varphi(w^k, v') = v \). Repeating this argument implies \( \varphi(w^k, v') = v \) for all \( w \in [v, u, \rightarrow) \).

Lemma 3.2: There exists an \( \alpha \in (0, \pi) \) such that for all different \( u, v, x \in \mathbb{R}^2 \) with \( \angle(x, u, v) < \alpha \) we have that \( \varphi(u, x) = u \) if \( \varphi(u, v) = u \).

Proof: It is a well-known fact from elementary topology (cf. Kreyszig (1978), p. 75) that all norms on \( \mathbb{R}^2 \) are equivalent. So every norm is equivalent to the Euclidean norm \( \| \cdot \|_E \), which implies that there exist \( \lambda_1, \lambda_2 > 0 \) with for all \( x \in \mathbb{R}^2 \):

\[
\frac{1}{\lambda_1} \| x \|_E \leq \delta(0, x) \leq \frac{1}{\lambda_2} \| x \|_E.
\]

Let \( \alpha = \arctan \frac{\lambda_2}{\lambda_1} \). Let different \( u, v, x \in \mathbb{R}^2 \), with \( \angle(x, u, v) < \alpha \) and \( \varphi(u, v) = u \). W.l.o.g. \( \delta(u, u) = 1 \) (cf. lemma 3.1). (See figure 1.)

![Fig. 1. Proof of lemma 3.2](image-url)
Let $y \in (u, x, \to)$ be such that $\|y - v\|_2 < \delta$. Then $\delta(y, v) < 1 = \delta(u, u)$. Let $z \in [y, u)$ be arbitrary. Let $\lambda \in (0, 1]$ be such that $z = \lambda y + (1 - \lambda)u$. Then

$$\delta(v, z) = \|\lambda y + (1 - \lambda)u - v\| \leq \|\lambda(y - v)\| + \|(1 - \lambda)(u - v)\|$$

$$= |\lambda| \|y - v\| + |1 - \lambda| \|u - v\|$$

$$< |\lambda| + |1 - \lambda| = 1 = \delta(v, u).$$

By SP we now have $\varphi(u^k, y') \neq z$ for all $z \in [y, u)$. PO implies $\varphi(u^k, y') \in [y, u]$, hence $\varphi(u^k, y') = u$. So by lemma 3.1: $\varphi(u^k, x') = u^k$.

The following lemma follows from repeated application of lemma 3.2.

**Lemma 3.3:** Let $u, v \in \mathbb{R}^2$ be such that $u \neq v$ and $\varphi(u^k, v') = u$. Then for all $x \in \mathbb{R}^2$: $\varphi(u^k, x') = u$.

**Lemma 3.4:** Let $u, v \in \mathbb{R}^2$ be such that $u \neq v$. Then $\varphi(u^k, v') = u$ if and only if $k \geq l$.

**Proof:** (i) Let $k \geq l$, $w = \varphi(u^k, v')$. Assume $w \neq u$. By lemma 2.3 we have $\varphi(u^k, w') = w$. Hence by lemma 3.3:

$$\varphi(x^k, w') = w \quad \text{for all } x \in \mathbb{R}^2. \quad (3.5)$$

Now let $a, b \in \mathbb{R}^2$ be such that $a \neq b$, $w \notin [a, b]$. Let $c = \varphi(a^k, b')$. Since $\varphi$ satisfies PO, $c \neq w$ (because $w \notin [a, b]$). Suppose $c \neq b$. By lemma 2.3 $\varphi(c^k, b') = c$. Lemma 3.3 implies $\varphi(c^k, x') = c$ for all $x \in \mathbb{R}^2$. Hence $\varphi(c^k, w') = w$ and $\varphi(c^k, x') = c$. This contradicts $w \neq c$. Hence $b = \varphi(a^k, b')$. Since $k \geq l$, by lemma 2.3 this implies $b = \varphi(a^l, b^k)$. By lemma 3.3: $\varphi(b^k, x') = b$ for all $x \in \mathbb{R}^2$, in particular $\varphi(b^k, w') = b$. So by (3.5), $b = w$, which contradicts $w \notin [a, b]$. Consequently, $\varphi(u^k, v') = w = u$.

(ii) Let $k < l$. By (i) $\varphi(u^k, v') = v \neq u$.

We are now in a position to prove theorem 2.2. This theorem states that, under the assumptions in this section, the number of agents $n$ must be odd.

**Proof of Theorem 2.2:** Suppose $n$ is even. Let $k = \frac{1}{2}n$. Let $u, v \in \mathbb{R}^2$ be such that $u \neq v$. Then by lemma 3.4 we have $\varphi(u^k, v') = u$ and $\varphi(u^k, v') = v$, which is impossible if $u \neq v$. \qed
4 Three Locations

In this section we will investigate the case where all preference vectors \( p(i) \) are concentrated in three locations. In other words, we will study \( \varphi(x^k, y^l, z^m) \), where \( x, y, z \in \mathbb{R}^2 \). Throughout this section we assume \( k, l, m \in \mathbb{N} \) to be such that \( k + l + m = n \), where \( n \) is odd. Furthermore, we assume \( \varphi \) to satisfy SP, AN and PO with respect to the metric \( \delta \) derived from the strictly convex norm \( \| \cdot \| \).

Lemma 4.1: Let \( k > l + m, x, y, z \in \mathbb{R}^2 \). Then \( \varphi(x^k, y^l, z^m) = x \).

Proof: Let \( w = \varphi(x^k, y^l, z^m) \). By SP, \( \varphi(x^k, w^{l+m}) = w \). So by lemma 3.4, \( w = x \). \( \square \)

In the remainder of this section we will assume \( k, l, m < \frac{1}{2} n \). We will next commence a long series of lemmas, which will eventually lead to a generalized median solution for the case of three locations (theorem 4.1). The first part of this series culminates mainly in lemma 4.12, in which for the first time orthogonal directions are established.

Lemma 4.2: Let \( z \in [x, y] \). Then \( \varphi(x^k, y^l, z^m) = z \).

Proof: Let \( w = \varphi(x^k, y^l, z^m) \). By lemma 3.4 we have: \( \varphi(x^k, z^l, z^m) = z \) and \( \varphi(z^k, y^l, z^m) = z \). Hence by SP: \( \delta(x, w) \leq \delta(x, z) \) and \( \delta(y, w) \leq \delta(y, z) \). On the other hand, by PO, \( w \in [x, y] \). This is only possible if \( w = z \). \( \square \)

Lemma 4.3: Let \( w = \varphi(a^k, b^l, c^m), x \in A(w, b, c) \setminus \{w\} \). Then \( \delta(a, w) < \delta(a, x) \).

Proof: (See figure 2.) Let \( y \in [a, x] \cap ([b, w] \cup [c, w]) \). Hence, \( \delta(a, y) \leq \delta(a, x) \). If \( y = w \), then we are done. Next assume \( y \neq w \). W.l.o.g. assume \( y \in [b, w] \). Let

![Fig. 2. Proof of lemma 4.3](image)
u = \frac{1}{2}y + \frac{1}{2}w. Hence u \in [y, w] = [b, w]. By SP, φ(a^4, b^1, w^m) = w. Lemma 4.2 implies φ(u^4, b^1, w^m) = u. So by SP δ(a, w) ≤ δ(a, u). By lemma 2.4, δ(a, w) < δ(a, y). Hence, δ(a, w) < δ(a, x).

Lemma 4.4: Let a, b, c, x, w ∈ R^2 be such that a, b, c are noncollinear, φ(a^4, b^1, c^m) = a and φ(a^4, x^1, c^m) = w. Then [c, w] ∩ (a, b, →) = ∅.

Proof: Suppose [c, w] ∩ (a, b, →) ≠ ∅. Let z ∈ [c, w] ∩ (a, b, →). By φ(a^4, x^1, c^m) = w and lemma 4.3, δ(c, w) ≤ δ(c, a). Hence,

δ(c, z) ≤ δ(c, a).  \hspace{1cm} (4.6)

On the other hand, φ(a^4, b^1, c^m) = a. Hence by lemma 4.3, δ(c, a) < δ(c, z). This contradicts (4.6).

Lemma 4.5: Let a, b, c ∈ R^2 be such that φ(a^4, b^1, c^m) = a. Let d ∈ [c, b, →) \ [b, c]. Then φ(a^4, d^1, c^m) = a.

Proof: If a, b, c are collinear the desired statement is a direct consequence of lemma 4.2. Now consider the case where a, b, c are not collinear. (See figure 3.)

Let v = φ(a^4, d^1, c^m). By lemma 4.4 it follows that v ∈ A(a, d, b) \ {a}. Consequently, by PO, v ∈ A(a, b, c) \ [b, a]

Let [w] = [d, v] ∩ [b, a]. By lemma 2.3 and φ(a^4, d^1, c^m) = v, we have φ(a^4, w^1, c^m) = v. By lemma 2.3 and φ(a^4, b^1, c^m) = a, we have φ(a^4, w^1, c^m) = a. Hence a = v.

The next three lemmas establish continuity properties of φ.

![Fig. 3. Proof of lemma 4.5](image-url)
Lemma 4.6: Let \( b, c \in \mathbb{R}^2 \) and \( g: \mathbb{R}^2 \to \mathbb{R} \), \( g(x) := \delta(x, \varphi(x^4, b^1, c^m)) \). Then \( g \) is continuous on \( \mathbb{R}^2 \).

Proof: Suppose \( g \) is not continuous on \( \mathbb{R}^2 \). So we can find some \( x, x(1), x(2), \ldots \in \mathbb{R}^2 \) such that \( x(i) \to x \) and \( g(x(i)) \not\to g(x) \).

We distinguish two cases:

(i) \( \lim_{i \to \infty} g(x(i)) > g(x) \),

(ii) \( \lim_{i \to \infty} g(x(i)) < g(x) \).

Case (i): Let \( \varepsilon < \frac{1}{2}(g(x) - g(x(i))) \). Let \( i \in \mathbb{N} \) be such that \( g(x(i)) - g(x) > \varepsilon \).

Then \( \delta(x(i), \varphi(x(i), b^1, c^m))) > \delta(x, \varphi(x^4, b^1, c^m)) + \varepsilon \)

\[ \geq \delta(x(i), \varphi(x^4, b^1, c^m)) \]

which contradicts SP.

Case (ii): Analogous to case (i). \( \square \)

Lemma 4.7: Let \( a, b, c, b(1), b(2), \ldots \in \mathbb{R}^2 \) be such that \( \varphi(a^k, b^1, c^m) = a \) and \( b(i) \to b \). Then \( \varphi(a^k, b(i), c^m) \to a \).

Proof: (Cf. Figure 4.) Let \( w \) be an accumulation point of \( \{w(i)\} \), where \( w(i) := \varphi(a^k, b(i), c^m) \). Lemma 4.6 implies \( \delta(b, w) = \delta(b, a) \). Suppose \( w \neq a \). Let \( z = \frac{1}{2}a + \frac{1}{2}w \). Since \( \delta \) satisfies strict convexity, we have by lemma 2.4: \( \delta(b, z) < \frac{1}{2}(\delta(b, a) + \delta(b, w)) \).

\[ \delta(w, w(i)) < \frac{1}{2}(\delta(b, w) - \delta(b, z)) \]

Fig. 4. Proof of lemma 4.7
Let $v \in [a, w(f)]$ be such that $[z, v]//[w, w(f)]$. Then $\delta(z, v) \leq \delta(w, w(f)) < \frac{1}{2} \delta(b, a) - \delta(b, z)$.

Hence, \[
\delta(b(f), v) \leq \delta(b(f), b) + \delta(b, z) + \delta(z, v)
\]
\[
< \frac{1}{2} \delta(b, a) - \delta(b, z) + \delta(b, z)
\]
\[
= \delta(b, w) - \frac{1}{2} \delta(b, w) - \delta(b, z)
\]
\[
< \delta(b, w) - \delta(w, w(f)) - \delta(b, b(f))
\]
\[
\leq \delta(b(f), w(f)).
\] (4.7)

On the other hand by SP, $\varphi(a^k, b(f), w(f^m)) = w(f)$, while because of lemma 4.2, $\varphi(a^k, v^j, w(f^m)) = v$. So by SP we have $\delta(b(f), w(f)) \leq \delta(b(f), v)$, which contradicts (4.7).

Consequently, $w = a$. Hence, $a$ is the only accumulation point of $\{w(i)\}$. Since $b(i) \to b$, we can define $\rho \in \mathbb{R}$ such that $a, b(i), c \in \{x \in \mathbb{R}^2 | \delta(b, x) \leq \rho\}$ for all $i \in \mathbb{N}$. So by PO, $w(i) \in \{x \in \mathbb{R}^2 | \delta(b, x) \leq \rho\}$ for all $i \in \mathbb{N}$. Since this set is compact, we can conclude $w(i) \to a$. \qed

Lemma 4.8: Let $a, b, c, a(1), a(2), \ldots, w \in \mathbb{R}^2$ with $\varphi(a^k, b^l, c^m) = w$ and $a(i) \to a$. Then $\varphi(a(i)^k, b^l, c^m) \to w$.

Proof: Let $v$ be an accumulation point of $\{v(i)\}$, where $v(i) := \varphi(a(i)^k, b^l, c^m)$. Lemma 4.6 implies $\delta(a, v) = \delta(a, w)$. Suppose $\delta(b, w) < \delta(b, v)$. Let $w(i) := \varphi(a(i)^k, w^l, c^m)$ for all $i$. Lemma 4.7 implies $w(f) \to w$. Let $f \in \mathbb{N}$ be such that $\delta(v, w(f)) < \frac{1}{2} (\delta(b, v) - \delta(b, w))$ and $\delta(w, w(f)) < \frac{1}{2} (\delta(b, v) - \delta(b, w))$. Hence, \[
\delta(b, w(f)) \leq \delta(b, w) + \delta(w, w(f))
\]
\[
< \delta(b, w) + \frac{1}{2} (\delta(b, v) - \delta(b, w))
\]
\[
= \delta(b, v) - \frac{1}{2} (\delta(b, v) - \delta(b, w))
\]
\[
< \delta(b, v) - \delta(v, w(f))
\]
\[
\leq \delta(b, v(f)).
\]

---

5 "//" means "is parallel to"; a one-point set is parallel to any segment.
On the other hand, by SP we have $\delta(b, v(f)) \leq \delta(b, w(f))$, a contradiction. Hence $\delta(b, v) \leq \delta(b, w)$. In a similar way we can prove $\delta(c, v) \leq \delta(c, w)$. Suppose $v \neq w$. Let $z = \frac{1}{2}v + \frac{1}{2}w$. Lemma 2.4 implies $\delta(a, z) < \delta(a, w)$, $\delta(b, z) < \delta(b, w)$, and $\delta(c, z) < \delta(c, w)$. This contradicts the Pareto optimality of $\varphi$. 

Lemma 4.9: Let $a, b, c, d \in \mathbb{R}^2$ be such that $\varphi(a^k, b^l, c^m) = a$ and $d \in [a, b, \rightarrow)$. Then $\varphi(a^k, d^l, c^m) = a$.

Proof: Let $w = \varphi(a^k, d^l, c^m)$. Suppose $w \neq a$. Lemma 4.2 implies that $a, b$ and $c$ are noncollinear. Lemma 2.4 implies $d \notin [a, b]$, hence $d \in [a, b, \rightarrow) \backslash [a, b]$. Let $u \in [c, b, \rightarrow) \backslash [b, c]$. Lemma 4.5 implies $\varphi(a^k, u^l, c^m) = a$. Let $x(\delta) = u + \lambda(d - u)$ for all $\lambda \in [0, 1]$. Let $y(\lambda) = \varphi(a^k, x(\delta), c^m)$. Hence, $y(0) = a, y(1) = w$. (See figure 5.)

Lemma 4.4 implies $[a, y(\lambda)] \cap (a, b, \rightarrow) = \emptyset$ for all $\lambda \in [0, 1]$. Hence by PO, $y(\lambda) \in A(a, c) \cup [d, a]$ for all $\lambda \in [0, 1]$. Consequently, $[x(\delta), y(\lambda)] \cap [a, d] \neq \emptyset$ for all $\lambda \in [0, 1]$. Let:

$$\alpha = \sup \{\lambda \in [0, 1] | y(\lambda) = a\}.$$  

Lemma 4.8 implies $y(\alpha) = a$. Since $y(1) = w$, this implies $\alpha < 1$. Let $\lambda_i \in (\alpha, 1]$ be such that $\lambda_i \downarrow \alpha$. Lemma 4.8 implies $y(\lambda_i) \to a$. Furthermore, $x(\lambda_i) \rightarrow x(\alpha)$. Consequently, we can take $j \in \mathbb{N}$ such that $[x(\lambda_j), y(\lambda_j)] \cap [b, a] \neq \emptyset$. Let $z \in [x(\lambda_j), y(\lambda_j)] \cap [b, a]$ for such a $j \in \mathbb{N}$. So by lemma 2.3, $\varphi(a^k, z^l, c^m) = a$ and $\varphi(a^k, z^l, c^m) = y(\lambda_j)$. Hence, $y(\lambda_j) = a$, whereas $\lambda_j > \alpha$. This contradicts (4.8).

Lemma 4.10: Let $a, b, c \in \mathbb{R}^2$ be noncollinear and $w \in A(a, b, c)$ such that $\varphi(a^k, b^l, w^m) = \varphi(a^k, w^l, c^m) = \varphi(w^k, b^l, c^m) = w$. Then $\varphi(a^k, b^l, c^m) = w$. 

\[
\text{Fig. 5. Proof of lemma 4.9}
\]
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Proof: Let \( v = \varphi(a^k, b^l, c^m) \). Then SP implies \( \delta(u, v) \leq \delta(u, w) \) for all \( u \in \{a, b, c\} \). Suppose \( v \neq w \). Let \( r = \frac{1}{2}v + \frac{1}{2}w \). Then by lemma 2.4:

\[
\delta(u, r) < \delta(u, w) \quad \text{for all } u \in \{a, b, c\} .
\] (4.9)

On the other hand, \( w \in A(a, b, c) \). By lemma 2.1 this contradicts (4.9).

Lemma 4.11: Let \( a, b, c, d, w \in \mathbb{R}^2 \) be such that \( \varphi(a^k, b^l, c^m) = w, d \in [w, a, \rightarrow) \). Then \( \varphi(d^k, b^l, c^m) = w \).

Proof: Lemma 2.3 implies:

\[
\varphi(w^k, b^l, c^m) = w .
\] (4.10)

Furthermore this lemma implies \( \varphi(a^k, w^l, c^m) = w \). Hence, by lemma 4.9:

\[
\varphi(d^k, w^l, c^m) = w .
\] (4.11)

Similarly:

\[
\varphi(d^k, b^l, w^m) = w .
\] (4.12)

By lemma 4.10, (4.10), (4.11) and (4.12) imply \( \varphi(d^k, b^l, c^m) = w . \) □

In the next lemma, a pair of orthogonal directions determined by the solution \( \varphi \) is identified.

Lemma 4.12: Let \( a, b, c, d \in \mathbb{R}^2 \) be such that \( \varphi(a^k, b^l, c^m) = d \in (b, c), a \neq d \). Then \( b - c \) and \( a - d \) are \( \| - \| - \text{orthogonal}. \)

Proof: In view of lemma 4.2, \( a, b \) and \( c \) cannot be collinear. (See figure 6.) From lemma 4.11 we obtain:

\[
\forall x \in (d, b, \rightarrow), \quad \forall y \in (d, c, \rightarrow), \quad \forall z \in [d, a, \rightarrow): \varphi(x^k, x^l, y^m) = d .
\] (4.13)

Hence, by lemma 4.3:
Fig. 6. Proof of lemma 4.12

\[ \forall x \in \langle \leftarrow, b, c, \rightarrow \rangle \setminus \{d\}, \quad \forall z \in (d, a, \rightarrow): \delta(z, d) < \delta(z, x). \]  
(4.14)

Let \( \bar{z} \in [a, d, \rightarrow] \setminus [a, d], \bar{x} \in (\leftarrow, b, c, \rightarrow). \) By (4.14):

\[ \delta(2d - \bar{z}, d) = \delta(\bar{z}, d) < \delta(\bar{z}, \bar{x}) = \delta(2d - \bar{z}, 2d - \bar{x}). \]  
(4.15)

(4.13) together with lemma 4.3 implies:

\[ \forall x \in \langle \leftarrow, b, c, \rightarrow \rangle \setminus \{d\}, \quad \forall z \in (d, a, \rightarrow): \delta(x, d) < \delta(x, z). \]  
(4.16)

Let \( \bar{x} \in [a, d, \rightarrow] \setminus [a, d], \bar{x} \in (\leftarrow, b, c, \rightarrow). \) By (4.16):

\[ \delta(2d - \bar{x}, d) = \delta(\bar{x}, d) < \delta(\bar{x}, \bar{z}) = \delta(2d - \bar{x}, 2d - \bar{z}). \]  
(4.17)

In view of (4.15) and (4.17), we conclude that \( b - c \) and \( a - d \) are \( \| \cdot \| \)-orthogonal. \( \square \)

Under the conditions of lemma 4.12, a pair of orthogonal directions is identified. Roughly, two things have to be proved: first, the actual occurrence of a situation as in lemma 4.12 (in a nontrivial way) has to be established; second, it has to be shown that the derived orthogonal directions indeed determine \( \varphi \) as a generalized median solution. The first part is the most tedious one and is only achieved in lemma 4.24 below. The second part involves, among other things, showing that \( \varphi \) is "translation covariant".
The next long series of lemmas establishes both goals, but the construction, unfortunately, is not easy to summarize in a few lines. The section “culminates” in theorem 4.1, establishing the characterization result (theorem 2.1) for the case of three locations.

**Lemma 4.13:** Let \( a, b, c \in \mathbb{R}^2 \) be such that \( \varphi(a^k, b^l, c^m) = a \). Let \( u \in (a, b) \) be such that \( v = \varphi(u^k, z^l, c^m) \neq u \). Let \( \tilde{b} \in [u, b, \rightarrow) \), \( \tilde{c} \in [a, c, \rightarrow) \). Then \( \varphi(u^k, \tilde{b}^l, \tilde{c}^m) \in [u, u, \rightarrow) \).

**Proof:** By lemma 4.2, \( a, b \) and \( c \) are noncollinear. Let \( w = \varphi(u^k, \tilde{b}^l, \tilde{c}^m) \). Suppose \( w \notin [u, v, \rightarrow) \). Lemma 4.11 implies:

\[
\varphi(a^k, \tilde{b}^l, \tilde{c}^m) = a.
\] (4.18)

By PO, \( v \in \partial(u, b, c) \). If \( v \in [b, u) \), then by lemma 4.11, \( \varphi(a^k, b^l, c^m) = v \), so \( v = a = u \), a contradiction. Hence, \( v \notin [b, u) \). Similarly \( w \notin [\tilde{b}, u) \). W.l.o.g. assume \( \angle(a, u, v) < \angle(a, u, w) \) (if not, we can use a similar argument). (See figure 7.)

Let \( y \in [w, u, \rightarrow) \) be such that \( (y, a)/[u, v) \). By lemma 4.11, \( \varphi(y^k, \tilde{b}^l, \tilde{c}^m) = w \). So by (4.18) and SP, \( \delta(y, w) < \delta(y, a) \). Hence,

\[
\| y - u \| < \| y - a \|.
\] (4.19)

Let \( z \in [u, u, \rightarrow) \) be such that \( (z, b)/[u, w) \). By lemma 4.11, \( \varphi(z^k, b^l, c^m) = v \). So by lemma 4.3, \( \delta(z, v) < \delta(z, b) \). Hence,

\[
\| z - u \| < \| z - b \|.
\] (4.20)

![Fig. 7. Proof of lemma 4.13](image-url)
Let $\lambda \in (0, \infty)$ be such that $z - b = \lambda(y - u)$. Then $z - u = \lambda(y - a)$. So (4.20) implies $\|y - a\| < \|y - u\|$, which contradicts (4.19).

\[\square\]

**Lemma 4.14:** Let $a, b, c \in \mathbb{R}^2$ be such that $\varphi(a^k, b^l, c^n) = a$. Let $u \in (a, b)$ be such that $\varphi(u^t, b^l, c^n) = w \neq u$. Then there are $y \in (u, b, \rightarrow)$, $z \in (a, c, \rightarrow)$ such that $w \in (y, z)$ and $\varphi(u^t, y^t, z^n) = w$.

**Proof:** Let $z(\lambda) = a + \lambda(c - a)$, $\lambda \in [0, 1]$. Let $f: [0, 1] \rightarrow \mathbb{R}^2$ be such that $f(\lambda) = \varphi(u^t, b^l, z(\lambda)^n)$. In view of lemma 4.8, $f$ is continuous on $[0, 1]$. $f(0) = u$ because of lemma 4.2; $f(1) = w$. Furthermore, by lemma 4.13, $f(\lambda) \in [u, w, \rightarrow)$ for all $\lambda \in [0, 1]$. Let $\lambda_{\text{min}} = \inf\{\lambda \in [0, 1] | f(\lambda) = w\}$. Because of the continuity of $f$, $f(\lambda_{\text{min}}) = w$. Hence, $\lambda_{\text{min}} > 0$. Let $z = z(\lambda_{\text{min}})$. Let $y(\mu) = u + \mu(b - u)$, $\mu \in [0, 1]$. Let $g: [0, 1] \rightarrow \mathbb{R}^2$ be such that $g(\mu) = \varphi(u^t, y(\mu)^t, z^n)$. In view of lemma 4.8, $g$ is continuous on $[0, 1]$. $g(0) = u$ because of lemma 3.4; $g(1) = w$. Furthermore, by lemma 4.13, $g(\mu) \in [u, w, \rightarrow)$ for all $\mu \in [0, 1]$. Let $\mu_{\text{min}} = \inf\{\mu \in [0, 1] | g(\mu) = w\}$. Because of the continuity of $g$, $g(\mu_{\text{min}}) = w$. Hence, $\mu_{\text{min}} > 0$. Let $y = y(\mu_{\text{min}})$. Then $[y, z] \cap [u, w, \rightarrow) \neq \emptyset$. Let $d \in [y, z] \cap [u, w, \rightarrow)$. If $d \neq [u, w, \rightarrow)$ Then $w \neq d$. Suppose $d \neq w$. Then the strict convexity of $d$ implies $\delta(y, w) + \delta(z, w) > \delta(y, z) = \delta(y, d) + \delta(z, d)$. W.l.o.g. assume $\delta(y, d) < \delta(y, w)$. Hence there is some $\beta \in [0, \mu_{\text{min}}]$ such that $\delta(y, \beta, d) < \delta(y, \beta, w)$.

\[
\delta(y, \beta, d) < \delta(y, \beta, w) .
\]

Suppose $g(\beta) \notin [u, w)$. Then, by the continuity of $g$ there has to be some $\mu \in [0, \beta]$ such that $g(\mu) = w$. This contradicts the definition of $\mu_{\text{min}}$. Consequently $g(\beta) \in [u, w)$. Hence, by SP, $\delta(y, \beta, \mu) < \delta(y, \beta, w)$. Since $w \in (g(\beta), d)$, by lemma 2.4: $\delta(y, \beta, g(\beta)) < \delta(y, \beta, d)$. So by lemma 2.4: $\delta(y, \beta, w) < \delta(y, \beta, d)$. This contradicts (4.21).

\[\square\]

**Lemma 4.15:** Let $a, b, c, d, w, u \in \mathbb{R}^2$ be such that $d \in (b, c)$, $a, b, c$ non-collinear, $u \in [b, c]$, $w \in A(a, b, c) \backslash [b, c]$, $w \notin A(a, b, c) \backslash [b, c]$, and $\varphi(a^k, b^l, c^n) = d$. Then $\varphi(w^k, b^l, c^n) = u$.

**Proof:** Let $\hat{y} = \varphi(w^k, b^l, c^n)$. (See figure 8.) Suppose $\hat{y} \neq u$. Because of the lemmas 4.2, 4.3 and 4.12 this implies $\hat{y} \notin [b, c]$. Hence, PO implies $\hat{y} \in A(a, b, c) \backslash [b, c]$. W.l.o.g. assume $\hat{y} \in [b, d]$. Let $n \in [b, d]$ be such that $(\hat{y}, \hat{u}) \notin (a, d)$. W.l.o.g. assume $u \in [b, d]$. Let $y = \varphi(\hat{w}^k, b^l, \hat{u}^m)$. Because of the lemmas 4.3 and 4.12 we have $y \notin [\hat{w}, \hat{u}]$. Furthermore by PO, $y \in A(\hat{w}, b, d)$ If $y \notin [\hat{w}, d]$, then there are $\hat{w} \in (y, \hat{w}, \rightarrow) \cap [d, a, \rightarrow)$ and $\hat{u} \in (y, \hat{u}, \rightarrow) \cap [a, d, \rightarrow)$. In that case, by lemma 4.11, $\varphi(\hat{w}^k, b^l, \hat{u}^m) = y$. But
by lemma 4.11, \( \varphi(\hat{w}, b, d^m) = d \). So, by SP, \( \delta(\hat{u}, y) < \delta(\hat{u}, d) \). Hence, \( \delta(\hat{u}, \hat{u}) < \delta(\hat{a}, \hat{u}) \). This contradicts lemma 4.12. Consequently, \( y = \hat{u} \). By SP, \( \delta(c, \hat{w}) \leq \delta(c, \hat{a}) \), which contradicts lemma 4.12. Therefore \( \hat{w} = u \).

**Lemma 4.15:** Let \( a, b, c, d \in \mathbb{R}^2 \) such that \( \varphi(a, b, c) = d \). Then \( \varphi(a, b, c) = d \).

**Proof:** W.l.o.g. \( a, b, c \) are noncollinear. Suppose \( \varphi(d, b, c) = e \neq d \). W.l.o.g. \( d \in (a, b) \). Then, by lemma 4.14, there are \( \tilde{c} \in (a, c, \rightarrow) \) and \( \tilde{d} \in (d, b, \rightarrow) \) such that \( \varphi(d, \tilde{b}, \tilde{c}) = \tilde{d} \in \tilde{b}, \tilde{c} \). (See figure 9.)

Let \( x \in (b, \tilde{c}) \), s.t. \( (d, x) \in (a, c) \). By lemma 4.3, we have \( \delta(b, a) < \delta(b, z) \) for all \( z \in (a, \tilde{c}) \). Since \( \tilde{d} \) is based on a norm we have that \( \delta(b, d) < \delta(b, z) \) for all \( z \in (\tilde{d}, x) \). Hence \( \tilde{d} \in (b, x) \). Consequently, there is an \( \tilde{a} \) and \( \tilde{d} \) such that \( \tilde{a} \in (a, \tilde{d}) \) and \( \tilde{d} \in (\tilde{b}, \tilde{c}) \) and \( (a, d) \in (b, e) \) and \( \tilde{a} \in (d, c) \). Hence, \( \varphi(\tilde{a}, \tilde{b}, \tilde{c}) = \tilde{d} \) by lemma 4.15. Hence, \( \varphi(a, b, c) = d \) by lemma 4.11. But \( \varphi(a, b, c) = a \) because of \( \varphi(a, b, c) = a \) and lemma 4.11.

**Lemma 4.16:** Let \( a, b, c, d \in \mathbb{R}^2 \) such that \( \varphi(a, b, c) = a \) and \( d \in \Delta(a, b, c) \). Then \( \varphi(a, b, c) = d \).

**Proof:** By lemma 4.2, w.l.o.g. \( a, b, c \) are noncollinear. Let \( w = \varphi(\tilde{a}, b, c) \). Suppose \( w \notin \tilde{a}, \tilde{c} \). Then \( w \notin \tilde{a}, \tilde{c} \) and \( \varphi(a, c) = \tilde{a} \neq \tilde{a} \). Let \( u \in \tilde{w}, \tilde{c} \) and \( \tilde{a}, \tilde{c} \). Lemma 4.11 implies \( \varphi(\tilde{a}, b, u) = w \). On the other hand, by lemma 4.11,
\[ \varphi(a^k, b^l, u^m) = a. \] Hence by lemma 4.16, \( \varphi(\bar{a}^k, b^l, u^m) = \bar{a}. \) So \( w = \bar{a}, \) a contradiction. Consequently, \( w \in [\bar{a}, \bar{c}]. \)

Now suppose \( w \neq \bar{a}. \) Hence by lemma 4.3, \( \delta(b, w) < \delta(b, \bar{a}). \) Let \( \bar{w} \in [b, w, \rightarrow) \cap [c, a]. \) This implies:

\[ \delta(b, \bar{w}) < \delta(b, a) \]

\( \text{(remember} \ \delta \ \text{is induced by a norm).} \)

On the other hand, \( \varphi(a^k, \bar{w}^l, c^m) = \bar{w}, \) whereas \( \varphi(a^k, b^l, c^m) = a. \) Hence by SP, \( \delta(b, a) \leq \delta(b, \bar{w}), \) which contradicts (4.22).

**Lemma 4.18:** Let \( \varphi(a^k, b^l, c^m) = a \) with \( a, b, c \in \mathbb{R}^2. \) Let \( \bar{a} \in (b, a, \rightarrow) \) and \( \bar{c} \in (b, c, \rightarrow) \) such that \( (\bar{a}, \bar{c})/(a, c). \) Then \( \varphi(\bar{a}^k, b^l, \bar{c}^m) = \bar{a}. \)

**Proof:** By lemma 4.2, w.l.o.g. \( a, b, \) and \( c \) are noncollinear. If \( \bar{a} \in [b, a], \) then \( \bar{c} \in [b, c] \) and thus \( \varphi(\bar{a}^k, b^l, \bar{c}^m) = \bar{a} \) because of lemma 4.17. Now assume \( \bar{a} \notin [b, a]. \) (See figure 10.) Let \( w = \varphi(\bar{a}^k, b^l, \bar{c}^m). \) By SP:

\[ w = \varphi(\bar{a}^k, b^l, w^m). \]

Suppose \( w \notin [b, \bar{a}]. \) Let \( v \in [w, b] \) be such that \( (a, v)/(\bar{a}, w). \) (4.23) implies by lemma 4.17: \( \varphi(a^k, b^l, v^m) = v. \) On the other hand, \( \varphi(a^k, b^l, c^m) = a, \) so by SP,
\[ \delta(c, a) \leq \delta(c, w). \] Hence, \( \delta(\bar{c}, \bar{a}) \leq \delta(\bar{c}, w). \) This contradicts lemma 4.3. Consequently, \( w \in [b, a]. \) By \( a = \varphi(a^k, b^1, c^m) \) and lemma 4.3, \( \delta(c, a) < \delta(c, x) \) for all \( x \in [b, a]. \) \( \delta \) is induced by a norm, hence \( \delta(\bar{c}, \bar{a}) < \delta(\bar{c}, x) \) for all \( x \in [b, a]. \) By lemma 3.4, \( \varphi(\bar{a}^k, b^1, \bar{a}^m) = \bar{a}. \) By SP, \( \delta(\bar{c}, w) \leq \delta(\bar{c}, \bar{a}), \) hence \( w = \bar{a}. \)

**Lemma 4.19:** Let \( a, b, c, u, \bar{a}, \bar{b}, \bar{c} \in \mathbb{R}^2 \) such that \( x = x + u \) for \( x \in \{a, b, c\}. \) Then \( \varphi(\bar{a}^k, b^1, \bar{a}^m) = \varphi(a, b, c) + u. \)

**Proof:** By lemma 4.10 it is sufficient to prove: If \( \varphi(a^k, b^1, c^m) = a, \) then \( \varphi(\bar{a}^k, b^1, \bar{a}^m) = \bar{a}. \) This follows evidently from lemma 4.18.

**Lemma 4.20:** Let noncollinear \( a, b, c \in \mathbb{R}^2 \) be such that \( \varphi(a^k, b^1, c^m) = d, \) where \( d \in (b, c). \) Then \( \varphi(a^k, c^1, b^m) = d. \)

**Proof:** Let \( w = \varphi(a^k, c^1, b^m). \) W.l.o.g. \( l \leq m \) (for \( l > m \) a similar proof is derived by interchanging the roles of \( b \) and \( c \)). Suppose \( w \in [b, c] \setminus \{d\}. \) Since \( \varphi(a^k, b^1, c^m) = d, \) by lemma 4.3:

\[ \delta(a, d) \leq \delta(a, w). \quad (4.24) \]

On the other hand, by lemma 4.2, \( \varphi(d^k, c^1, b^m) = d. \) So SP implies \( \delta(a, w) \leq \delta(a, d). \) This contradicts (4.24). Consequently, \( w \notin [b, c] \setminus \{d\}. \)

Suppose \( w \neq d. \) Then by PO, \( w \in d(a, b, c) \setminus \{b, c\}. \) Let \( u \in \{b, c\} \) be such that \( (u, w) \) \( \parallel (d, a). \) So by lemma 4.15, \( \varphi(w^k, b^1, c^m) = u. \) On the other hand, by SP: \( \varphi(w^k, c^1, b^m) = w. \) Hence, by SP:

\[ \delta(b, w) \leq \delta(b, u). \quad (4.25) \]

Lemma 4.12 implies that the directions \( a - d \) and \( c - b \) are \( \| \cdot \| \)-orthogonal. Therefore the directions \( w - u \) and \( c - b \) are \( \| \cdot \| \)-orthogonal. Hence by lemma 2.6, \( \delta(b, u) < \delta(b, w). \) This contradicts (4.25).
Lemma 4.21: Let noncollinear \( a, b, c \in \mathbb{R}^2 \) be such that \( \varphi(a^k, b^l, c^m) = d \in (b, c) \). Let \( u \in [a, d, \rightarrow] \backslash [a, d] \). Then \( \varphi(a^k, u^l, c^m) = d \).

Proof: Let \( a(\lambda) = d + \lambda(a - d) \) for \( \lambda \in [0, \infty) \). (See figure 11.) Let \( w(\lambda) = \varphi(a(\lambda)^k, b^l, c^m) \) for \( \lambda \in [0, \infty) \). By lemma 4.11, \( \varphi(a(\lambda)^k, b^l, c^m) = d \). Hence by SP:

\[
\delta(u, w(\lambda)) \leq \delta(u, d) \quad \text{for all} \ \lambda \in [0, \infty) .
\] (4.26)

By PO, \( w(\lambda) \in \Delta(a(\lambda), u, c) \) for all \( \lambda \in [0, \infty) \). So lemma 4.12 implies \( w(\lambda) \in \Delta(d, u, c) \) for all \( \lambda \in [0, \infty) \). \( \Delta(d, u, c) \) is a compact set. Now let \( \lambda_j \to \infty \), \( \lambda_j = 1 \) and \( w(\lambda_j) \to \bar{w} \), with \( \bar{w} \in \Delta(d, u, c) \). Suppose \( w(1) \neq d \). Clearly, \( \bar{w} \neq d \), otherwise \( \delta(a, w(\lambda_j)) < \delta(a, w(1)) \) for \( \lambda_j \) large enough, contradicting SP. Suppose \( \bar{w} \in [c, d] \). Lemma 4.12 implies \( \delta(u, \bar{d}) < \delta(u, \bar{w}) \). Choose \( j \in \mathbb{N} \) such that \( \delta(w(\lambda_j), \bar{w}) < \delta(u, \bar{w}) - \delta(u, d) \). Hence, \( \delta(u, d) < \delta(u, w(\lambda_j)) \). This contradicts (4.26). Consequently, \( \bar{w} \notin [c, d] \).

So \( \bar{w} \in \Delta(u, d, c) \backslash [d, c] \). Let \( y \in [d, c] \) be such that \( (\bar{w}, y)//(u, d) \). Let \( z(\lambda_j) \in [d, c] \cap [w(\lambda_j), a(\lambda_j)] \). Hence \( z(\lambda_j) \to y \). Lemma 4.12 implies \( \delta(c, y) < \delta(c, \bar{w}) \). Choose \( j \in \mathbb{N} \) such that \( \delta(\bar{w}, w(\lambda_j)) < \frac{1}{2}(\delta(c, \bar{w}) - \delta(c, y)) \) and \( \delta(y, z(\lambda_j)) < \frac{1}{2}(\delta(c, \bar{w}) - \delta(c, y)) \). Hence:

\[
\delta(c, z(\lambda_j)) < \delta(c, w(\lambda_j)) .
\] (4.27)

On the other hand, by lemma 2.3, \( \varphi(a(\lambda_j)^k, w(\lambda_j)^l, c^m) = w(\lambda_j) \). Lemma 4.2 implies \( \varphi(a(\lambda_j)^k, w(\lambda_j)^l, z(\lambda_j)^m) = z(\lambda_j) \). Hence by SP, \( \delta(c, w(\lambda_j)) \leq \delta(c, z(\lambda_j)) \). This contradicts (4.27). Therefore \( w(1) = d \). So \( \varphi(a^k, u^l, c^m) = d \). \( \square \)
Lemma 4.22: Let a, b, c, d, u be such that d ∈ (b, c) and d ∈ (a, u) while a, b, c are noncollinear. Let x, y, z ∈ \{a, b, c, u\} be such that \#\{x, y, z\} = 3. Then \(\varphi(a^k, b^l, c^m) = d\) if and only if \(\varphi(x^k, y^l, z^m) = d\).

Proof: Since \(\varphi(a^k, b^l, c^m) = d\), \(\varphi(a^k, c^l, b^m) = d\) because of lemma 4.20. The desired result now can be obtained by repeated application of lemma 4.21.

Lemma 4.23: Let a, b, c, d, x ∈ \(\mathbb{R}^2\) with b, c ∈ \([a, d]\), \(\varphi(a^k, c^l, x^m) = x\), \(\varphi(b^k, d^l, x^m) = x\). Then \(\varphi(b^k, c^l, x^m) = x\).

Proof: Let z = \(\varphi(b^k, c^l, x^m)\). By \(\varphi(a^k, c^l, x^m) = x\) and SP:

\[
\delta(a, x) \leq \delta(a, z). \tag{4.28}
\]

By \(\varphi(b^k, d^l, x^m) = x\) and SP:

\[
\delta(d, x) \leq \delta(d, z). \tag{4.29}
\]

By PO, \(z \in A(b, c, x)\). Suppose \(z \neq x\). Then \(((x, z) \setminus ([x, z]) \cap [a, d]) \neq \emptyset\). Let u ∈ \(((x, z) \setminus ([x, z]) \cap [a, d])\). Hence, \(z \in [u, x]\). By lemma 2.4, (4.28) implies \(\delta(a, x) \leq \delta(a, u)\). Similarly by (4.29), \(\delta(d, x) \leq \delta(d, u)\). Hence, \(\delta(a, x) + \delta(x, d) = \delta(a, d)\).

So because of the strict convexity of \(\delta, x \in [a, d]\). Hence, by PO, (4.28) and (4.29), \(z = x\), a contradiction.

Lemma 4.24: There are noncollinear x, y, z ∈ \(\mathbb{R}^2\) such that \(\varphi(x^k, y^l, z^m) \in (y, z)\).

Proof: Take \(\bar{a}, b, c \in \mathbb{R}^2\) such that \(\bar{a}, b, c\) are noncollinear and \(\delta(\bar{a}, b) = \delta(\bar{a}, c)\). Let a = \(\varphi(\bar{a}^k, b^l, c^m)\). If a ∈ \([b, c]\), lemma 4.3 implies \(\delta(\bar{a}, a) < \delta(\bar{a}, b)\) or \(\delta(\bar{a}, a) < \delta(\bar{a}, c)\). Hence a ∉ \([b, c]\), so a ∈ (b, c) and we are done.

W.l.o.g., a ∉ \([b, c]\). By PO, a, b and c are noncollinear. SP implies \(\varphi(a^k, b^l, c^m) = a\). Let u(λ) = a + λ(a − c), λ ∈ [0, ∞). For λ large, \(\delta(b, u(λ)) < \delta(b, u(λ))\), and therefore by SP, \(\varphi(u(λ)^k, b^l, c^m) \neq u(λ)\).

Let \(\lambda_{\max} = \sup\{λ \in [0, ∞) | \varphi(u(λ)^k, b^l, c^m) = u(λ)\}\). Hence \(\lambda_{\max} < ∞\). Let d = u(\(λ_{\max}\)). Then by continuity (see lemma 4.8):

\[
\varphi(d^k, b^l, c^m) = d. \tag{4.30}
\]
By definition of $d$:
\[
\varphi(x^k, b^l, c^m) \neq x \quad \text{for all } x \in [c, d, \rightarrow) \setminus [c, d].
\] (4.31)

Let $v \in [c, d, \rightarrow) \setminus [c, d]$ be such that:
\[
\delta(v, d) < \delta(v, b).
\] (4.32)

Let $w = \varphi(v^k, b^l, d^m)$. Suppose $w \notin [b, d]$. (See figure 12.) Then by PO, $w \in d(v, b, d) \cap [b, d]$. Lemma 2.3 implies $\varphi(w^k, b^l, d^m) = w$. Choose $u \in [w, d, \rightarrow) \cap [w, d]$ arbitrarily. By lemma 4.11, $\varphi(w^k, b^l, u^m) = w$. Let $\tilde{b} \in [b, u]$ be such that $(\tilde{b}, d) \setminus [b, w]$. Let $\tilde{c} \in [d, c, \rightarrow) \cap [b, u]$. Then by lemma 4.18, $\varphi(\tilde{d}^k, \tilde{b}^l, u^m) = d$. On the other hand, (4.30) and lemma 4.11 imply $\varphi(\tilde{d}^k, b^l, z^m) = d$. So by lemma 4.6, $\varphi(\tilde{d}^k, \tilde{b}^l, \tilde{c}^m) = d$. Let $\tilde{d} \in [b, w] \cap [v, d)$. Then by lemma 4.18, $\varphi(\tilde{d}^k, \tilde{b}^l, \tilde{c}^m) = \tilde{d}$. So by lemma 4.11, $\varphi(\tilde{d}^k, b^l, c^m) = \tilde{d}$. This contradicts (4.31). Consequently, $w \in [b, d]$. By (4.32) and lemma 4.3, $w \neq b$, hence $w \in [d, b)$. If $w \neq d$, then $\varphi(v^k, b^l, d^m) = w \in (b, d)$, and we are done. Now assume $w = d$. By lemma 4.2, $\varphi(v^k, d^l, c^m) = d$. Hence, lemma 4.10 implies $\varphi(v^k, b^l, c^m) = d \in (v, c)$. So by lemma 4.22, $\varphi(b^k, v^l, c^m) = d \in (v, c)$.

Lemma 4.25: Let noncollinear $a, b, c \in \mathbb{R}^2$ be such that $\varphi(a^k, b^l, c^m) = d \in (b, c)$. Let $y \in (\leftarrow, a, d, \rightarrow), z \in (\leftarrow, b, c, \rightarrow)$. Then $\varphi(d^k, y^l, z^m) = d$.

Proof: Let $u = 2d - a$. W.l.o.g. $y \in [d, u, \rightarrow), z \in [d, b, \rightarrow)$. Lemma 4.22 implies $\varphi(a^k, u^l, b^m) = d$. So by lemma 4.11, $\varphi(d^k, y^l, z^m) = d$. 

\[\blacksquare\]
Lemma 4.26: There exist \( v, w \in \mathbb{R}^2, \lambda, \mu: \mathbb{R}^2 \to \mathbb{R} \) such that:

(i) \( x = \lambda(x) \cdot v + \mu(x) \cdot w \) for all \( x \in \mathbb{R}^2 \).
(ii) \( \varphi(x^k, y^l, z^m) = \text{med}(\lambda(x), \lambda(y), \lambda(z)) \cdot v + \text{med}(\mu(x), \mu(y), \mu(z)) \cdot w \) for all \( x, y, z \in \mathbb{R}^2 \).
(iii) \( v \) and \( w \) are \( \| \cdot \| \)-orthogonal.

Proof: By lemma 4.24 there are noncollinear \( a, b, c \in \mathbb{R}^2 \) such that \( \varphi(a^k, b^l, c^m) = d \in (b, c) \). Let \( v = c - d, w = a - d \), \( v \) and \( w \) are linearly independent, so we can define \( \lambda, \mu: \mathbb{R}^2 \to \mathbb{R} \) according to (i). Furthermore, by lemma 4.12, \( v \) and \( w \) are orthogonal, so (iii) is satisfied. Let \( x, y, z \in \mathbb{R}^2 \) be arbitrary. Let:

\[
\begin{align*}
    r &= \text{med}(\lambda(x), \lambda(y), \lambda(z)) \cdot v + \text{med}(\mu(x), \mu(y), \mu(z)) \cdot w.
\end{align*}
\]

Hence, by lemma 2.7:

\[
\begin{align*}
    r &\in d(x, y, z). \tag{4.33}
\end{align*}
\]

Furthermore, \( \lambda(r) = \text{med}(\lambda(r), \lambda(y), \lambda(z)) \) and \( \mu(r) = \text{med}(\mu(r), \mu(y), \mu(z)) \). Hence, \([y, z] \cap (\leftrightarrow, r + v, \rightarrow) \neq \emptyset\) and \([y, z] \cap (\leftrightarrow, r + w, \rightarrow) \neq \emptyset\). Let \( u \in [y, z] \cap (\leftrightarrow, r + v, \rightarrow) \) and \( \tilde{u} \in [y, z] \cap (\leftrightarrow, r + w, \rightarrow) \). If \( u = \tilde{u} \) then \( u = \tilde{u} = r \), so \( r \in [y, z] \), which implies by lemma 4.2:

\[
\begin{align*}
    \varphi(r^k, y^l, z^m) = r &\quad \text{if } u = \tilde{u}. \tag{4.34}
\end{align*}
\]

Suppose \( u \neq \tilde{u} \). Then lemma 4.19 implies \( \varphi((r + a - d)^k, (r + b - d)^l, (r + c - d)^m) = r \), where \( r \in (r + b - d, r + c - d) \) and \( r + a - d, r + b - d, r + c - d \) are noncollinear. Hence, by lemma 4.25, \( \varphi(r^k, u^l, \tilde{u}^m) = r \). Similarly \( \varphi(r^k, \tilde{u}^l, u^m) = r \), so by lemma 4.5:

\[
\begin{align*}
    \varphi(r^k, y^l, z^m) = r &\quad \text{if } u \neq \tilde{u}. \tag{4.35}
\end{align*}
\]

(4.34) and (4.35) together imply:

\[
\begin{align*}
    \varphi(r^k, y^l, z^m) = r. \tag{4.36}
\end{align*}
\]

Similarly we can prove:

\[
\begin{align*}
    \varphi(x^k, r^l, z^m) = r &\quad \text{and } \varphi(x^k, y^l, r^m) = r. \tag{4.37}
\end{align*}
\]

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By lemma 4.10, (4.33), (4.36) and (4.37) imply $\varphi(x^k, y^l, z^m) = r$. 

\[ \blacksquare \]

**Theorem 4.1:** Let $\delta$ be induced by a strictly convex norm $\| \cdot \|$. Let the number of agents be odd. Let $\varphi$ satisfy anonymity, Pareto optimality and strategy-proofness. Then there exist $a, b \in \mathbb{R}^2 \setminus \{0\}$, $\lambda, \mu : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that:

(i) $x = \lambda(x) \cdot a + \mu(x) \cdot b$ for all $x \in \mathbb{R}^2$.

(ii) For all $k, l, m \in \mathbb{N}$ such that $k + l + m = n$ and $k, l, m < \frac{1}{2} n$:

\[
\varphi(x^k, y^l, z^m) = \text{med}(\lambda(x), \lambda(y), \lambda(z)) \cdot a + \text{med}(\mu(x), \mu(y), \mu(z)) \cdot b
\]

for all $x, y, z \in \mathbb{R}^2$.

(iii) For all $k, l, m \in \mathbb{N}$ such that $k + l + m = n$:

\[
\varphi(x^k, y^l, z^m) = \begin{cases} 
  x & \text{if } k > \frac{1}{2} n \\
  y & \text{if } l > \frac{1}{2} n \\
  z & \text{if } m > \frac{1}{2} n
\end{cases}
\]

(iv) $a$ and $b$ are $\| \cdot \|$-orthogonal.

**Proof:** If $j > \frac{n}{2}$ for some $j \in \{ k, l, m \}$ the desired result follows from lemma 4.1.

Now assume $k, l, m < \frac{n}{2}$. By lemma 4.26 there exist $a, b \in \mathbb{R}^2 \setminus \{0\}$, $\lambda, \mu : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that:

\[
x = \lambda(x) \cdot a + \mu(x) \cdot b \quad \text{for all } x \in \mathbb{R}^2.
\]

\[
\varphi(x^k, y^{(n-1)/2}, z^{(n-1)/2}) = \text{med}(\lambda(x), \lambda(y), \lambda(z)) \cdot a + \text{med}(\mu(x), \mu(y), \mu(z)) \cdot b
\]

for all $x, y, z \in \mathbb{R}^2$.

$a$ and $b$ are $\| \cdot \|$-orthogonal. (4.38)

Let $\nu = \varphi(x^1, y^{(n-1)/2}, z^{(n-1)/2})$. Then by (4.38) we have $\varphi(a^1, b^{(n-1)/2}, c^{(n-1)/2}) = \nu$ for all $a, b, c$ with $\{a, b, c\} = \{x, y, z\}$. Hence by SP it follows $\varphi(a^1, b^{(n-1)/2}, c^{(n-1)/2}) = v$ for all $b \neq c$ with $b, c \in \{x, y, z\}$. Hence by SP it follows that $\varphi(a^k, b^l, c^m) = \nu$ for all $b \neq c$ with $b, c \in \{x, y, z\}$ and $\{k, l, m\} = \{k, l, m\}$. So by lemma 4.10 it follows that $\varphi(x^k, y^l, z^m) = v$. 

\[ \blacksquare \]
5 Proof of the Main Theorem

We are now sufficiently equipped to prove theorem 2.1. So let $\delta$ be induced by the strictly convex norm $\| \cdot \|$, let the number of agents $n$ be odd, and let $\phi: (\mathbb{R}^2)^N \to N$ be a solution. If $\phi$ is the generalized median solution with respect to the $\| \cdot \|$-orthogonal pair $\{a, b\}$, with $a, b \in \mathbb{R}^2\setminus\{0\}$, then lemma 2.7 implies that $\phi$ is anonymous, Pareto optimal and strategy-proof (with respect to $\delta$). Conversely, assume that $\phi$ satisfies these three properties. We will show that $\phi$ is the generalized median solution with respect to an $\| \cdot \|$-orthogonal pair $\{a, b\}$. In other words, we will prove that there exist nonzero $a, b \in \mathbb{R}^2, \lambda, \mu: \mathbb{R}^2 \to \mathbb{R}$ such that:

(i) $x = \lambda(x) \cdot a + \mu(x) \cdot b$ for all $x \in \mathbb{R}^2$.
(ii) $a$ and $b$ are $\| \cdot \|$-orthogonal.
(iii) $\phi(p) = \text{med}(\lambda(p(1)), \ldots, \lambda(p(n))) \cdot a + \text{med}(\mu(p(1)), \ldots, \mu(p(n))) \cdot b$ for all $p \in (\mathbb{R}^2)^N$.

Proof: Let $A(j) = \{ p \in (\mathbb{R}^2)^N | \# \{ p(i) | i \in N \} \leq j \}$. By theorem 4.1 and lemma 3.4 we can take nonzero $a, b \in \mathbb{R}^2, \lambda, \mu: \mathbb{R}^2 \to \mathbb{R}$ such that (i) and (ii) are satisfied, while $\phi(p) = \text{med}(\lambda(p(1)), \ldots, \lambda(p(n))) \cdot a + \text{med}(\mu(p(1)), \ldots, \mu(p(n)))$ for all $p \in A(3)$. Let $a, b \in \mathbb{R}^2, \lambda, \mu: \mathbb{R}^2 \to \mathbb{R}$ be defined accordingly.

Now assume:

$$\phi(p) = \text{med}(\lambda(p(1)), \ldots, \lambda(p(n))) \cdot a + \text{med}(\mu(p(1)), \ldots, \mu(p(n))) \cdot b$$

for all $p \in A(j)$,  \hspace{1cm} (5.39)

where $j \in \{3, \ldots, n - 1\}$. Let $p \in A(j + 1)$ be arbitrary. Let $w = \phi(p), \alpha = \lambda(w), \beta = \mu(w)$. Suppose $\alpha < \text{med}(\lambda(p(1)), \ldots, \lambda(p(n)))$. Let $\delta \in \mathbb{R}$ be such that $\alpha < \delta \leq \min \{ \lambda(p(i)) | \lambda(p(i)) > \alpha \}$. Let $q \in A(j + 1)$ be such that $q(i) = w$ if $\lambda(p(i)) \leq \alpha$, while $\lambda(q(i)) = \delta$ and $q(i) \in \{ w, p(i) \}$ if $\lambda(p(i)) > \alpha$. Lemma 2.3 implies $\phi(q) = w$.

Let $r \in A(j + 1)$ be such that $r(i) \in \{ q(i), w \}$, while $\# \{ i \in N | r(i) = w \} = \frac{n - 1}{2}$. Lemma 2.3 implies $\phi(r) = w$. Hence, $\lambda(\phi(r)) = \alpha < \delta = \text{med}(\lambda(r(1)), \ldots, \lambda(r(n)))$.

Because of (5.39), $r \notin A(j)$. Hence, $\# \{ r(i) | i \in N \land \lambda(r(i)) = \delta \} \geq 3$.

W.l.o.g. $\# \{ r(i) | i \in N \land \lambda(r(i)) = \delta \} \geq 3$. Let $k, l \in N$ be such that $\lambda(r(k)) = \lambda(r(l)) = \delta$, $\mu(r(k)) \geq \beta$. Let $s \in (\mathbb{R}^2)^N$ be such that $s(i) = r(k)$ if $r(i) = r(l)$ and $s(i) = r(l)$ if $r(i) \neq r(l)$. (See figure 13.) So $s \in A(j)$. Hence by (5.39), $\phi(s) = \delta \cdot a + \eta \cdot b$, with $\eta \in [\beta, \mu(r(k))]$. SP implies:

$$\delta(r(l), w) \leq \delta(r(l), \phi(s)).$$  \hspace{1cm} (5.40)
On the other hand, $a$ and $b$ are $\| \cdot \|$-orthogonal, hence in view of lemma 2.6:

$$\delta(r(l), w) > \delta(r(l), \varphi(s)).$$

This contradicts (5.40). Consequently, $a \geq \text{med}(\lambda(p(1)), \ldots, \lambda(p(n)))$. Similarly, one can prove $a \leq \text{med}(\lambda(p(1)), \ldots, \lambda(p(n)))$. Hence, $a = \text{med}(\lambda(p(1)), \ldots, \lambda(p(n)))$. Analogously, $\beta = \text{med}(\mu(p(1)), \ldots, \mu(p(n)))$. $p \in A(j + 1)$ was chosen arbitrarily, hence,

$$\varphi(p) = \text{med}(\lambda(p(1)), \ldots, \lambda(p(n))) \cdot a + \text{med}(\mu(p(1)), \ldots, \mu(p(n))) \cdot b$$

for all $p \in A(j + 1)$.

Repeated application of this argument implies (iii).

6 More than Two Dimensions

In this section we give an example of a strictly convex norm on $\mathbb{R}^3$ and a solution $\varphi: (\mathbb{R}^3)^N \to \mathbb{R}^3$, for $N = \{1, 2, 3\}$, which is anonymous and Pareto optimal and strategy-proof with respect to that norm; the definitions of these concepts are straightforward adaptations of the corresponding definitions for the two-dimensional case. A proof is available from the authors.

Let $\| \cdot \|_1: \mathbb{R}^3 \to \mathbb{R}$ be the sum norm, i.e., $\|x\|_1 = |x_1| + |x_2| + |x_3|$, and let $\| \cdot \|_2$ denote the Euclidean norm on $\mathbb{R}^3$. Let $\varepsilon \in (0, \frac{1}{2})$ be fixed. Then define $\| \cdot \|: \mathbb{R}^3 \to \mathbb{R}$ by $\|x\| = \|x\|_1 + \varepsilon \|x\|_2$ for all $x \in \mathbb{R}^3$. 
The announced solution \( \varphi \) is defined to be the generalized median solution with respect to the standard basis of \( \mathbb{R}^3 \). It is not hard to show that \( \varphi \) is not Pareto optimal w.r.t. the Euclidean norm \( \| \cdot \|_2 \); moreover, as remarked in the Introduction, a solution with the mentioned properties w.r.t. \( \| \cdot \|_2 \) does not exist, see PSS (1992).

7 Related Literature

Apart from Moulin (1980), most closely related to the present paper is the work by Kim and Roush (1984). For the case of the Euclidean norm, they characterize all solutions that are anonymous, strategy-proof, and continuous, for dimension \( m = 2 \) and at least two agents. They show that any such solution must be of the following form. Choose a pair of orthogonal axes, and \( n + 1 \) fixed points \( a^1, a^2, \ldots, a^{n+1} \in (\mathbb{R} \cup \{-\infty, \infty\})^2 \). To a profile \( p \in (\mathbb{R}^2)^N \), the solution assigns the point with the medians of the collection \( \{ p(i), a^j | i \in N, j = 1, 2, \ldots, n + 1 \} \) on these axes as coordinates. Kim and Roush (1984) further show that, if Pareto optimality is added as a requirement, then \( n \) must be odd, \( m \) must be equal to 2 (\( m \geq 2 \) is presupposed), and the solutions become exactly those identified in this paper or in PSS (1992). Thus, we have proved that the continuity requirement is redundant for these results. This is fortunate since, in our view, continuity is a mathematical requirement which, from an interpretational point of view, is less attractive than Pareto optimality.

Other directly related work includes Border and Jordan (1983), Bordes et al. (1990), Laffond (1980). Border and Jordan (1983), in a closely related model, derive "separability" of the solution from strategy-proofness and "unanimity". This is consistent with our results: generalized median solutions are separable (i.e., can be determined coordinatewise). Note that we too get separability as a consequence from more basic axioms, contrary to for instance Rubinstein and Fishburn (1986) who put it forward as a basic requirement. In PSS (1991) the work of Border and Jordan (1983) is extended.

In another strand of literature, location problems are viewed from a perspective of fairness. In Rosenmüller (1982), location problems are related to so-called nonsidepayment games, which, however, must be interpreted in a different way. Coalitions are not able to enforce their "worths" but, rather, these worths represent what would be "ideal" given that outside players were not present. In particular, if only the grand coalition and single-player coalitions are allowed, then the individual "worths" are the players' ideal points. It is assumed that a central planner knows all preferences; players cannot lie. Fair solutions can then be obtained by (cooperative) game-theoretical methods. A similar line of thought is pursued in Ostmann (1982), Ostmann and Straub (1979), Richter (1979), and Rosenmüller (1980).
Finally, we mention some papers which are related as to the choices of the domains of alternatives, and preferences: continua, and restricted domains of preferences, respectively. Among these are Chichilnisky and Heal (1981), Moreno and Walker (1990), Satterthwaite and Sonnenschein (1981), Sprumont (1989), Zhou (1990a, b). Important from a general, methodological, point of view is Barbera and Peleg (1990).

References

Ostmann A, Straub M (1979) On the Geometry behind the Fairness Concept of Rawls and Kolm. University of Bielefeld
Richter WP (1979) Shapley’s Value and Fair Solutions of Location Conflicts. In: Game Theory and Related Topics (ed. by Morschin/Pallaschke), North-Holland, Amsterdam
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