Extending Invariant Solutions

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In this paper we show that an invariant solution for normal form games can be extended uniquely to an invariant solution for strategic form games. This result has the following consequence for the reduction of a (normal form) game. Suppose that the pure strategies are removed that are payoff equivalent with some (possibly mixed) strategy. Then, if one is concerned with an invariant solution, a further reduction by identifying for each player arbitrary payoff-equivalent strategies is not necessary. Journal of Economic Literature Classification Number: C72.

1. INTRODUCTION

In 1986, Kohlberg and Mertens proposed to study mappings that assign to each normal form game a collection of closed, nonempty sets of strategy profiles. They called such a mapping a solution. Furthermore they formulated a number of properties such a solution should satisfy. One of these properties, invariance, plays a central role in this paper. Roughly speaking, a solution is invariant if the solution sets of a game do not change when the game is reduced. In this paper two ways of reducing a game are considered. The first way of reducing a game is eliminating iteratively pure strategies that are payoff equivalent with some (possibly mixed) strategy. When this method is applied to a normal form game, the result is again a normal form game. The second, more rigorous reduction method involves the identification of payoff-equivalent strategies for any player of a game irrespective of the question of whether the strategies identified were pure or not. In general, the result of this type of reduction is not a normal form game but a game in strategic form: the strategy spaces are polytopes, not necessarily simplices (even when we started with a normal form game). Thus we are forced to extend the concept of a solution to strategic form games.
In fact the two ways of reducing a game described above generate two types of invariance: (a relatively simple) one for solutions over normal form games and (a more complex) one for solutions over strategic form games. However, the solutions introduced in the literature are usually defined for normal form games only. Sometimes it is not clear how to extend the given definition to the larger class of strategic form games.

In this paper we will show that an invariant solution for normal form games can be extended uniquely to an invariant solution for strategic form games. This implies that the more complex form of invariance (over strategic form games) can be checked directly by investigating the simple form of invariance (over normal form games).

Note that this also solves the problem of extending (the definition of) an invariant solution for normal form games to the class of strategic form games: there is only one way to do the job.

In order to formalize some of the foregoing concepts, we make use of the following method—introduced by Mertens (1987) (see also van Damme, 1996)—to describe a reduction of a (strategic form) game. He identifies two strategic form games $\Gamma' = (P, v)$ and $\Gamma'' = (Q, w)$ if for any $i$ there exists an affine and surjective mapping $f_i: P_i \rightarrow Q_i$ such that $v_i = w_i \circ f_i$. We will call a solution weakly invariant if the solution sets of the game $\Gamma'$ are precisely the images under $f$ of the solution sets of the game $\Gamma$. A solution is called invariant if moreover the preimage of any solution set $S$ of the game $\Gamma''$ is the union of solution sets of the game $\Gamma'$ whose images under $f$ are $S$.

In Section 2, we define normal form games and strategic form games. In Section 3, Mertens' method of identifying two strategic form games is described. In Section 4, the (weak) invariance of a solution is defined. Furthermore our main result is proved. In the last section, the (weak) invariance of solutions for normal form games is considered. We show that one game is a reduction of another game if and only if the first game arises from the other by adding convex combinations of pure strategies as new pure strategies.

Notation. For $n \in \mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{R}^n$ is the vector space of $n$-tuples of real numbers. If $T$ is a finite set, $\Delta(T)$ is the set of probability distributions on $T$. The kernel of a linear map $A$ is denoted by $\text{Ker}(A)$. For a convex set $C$, $\text{ext}(C)$ is the set of extreme points of $C$.

2. GAMES IN STRATEGIC AND NORMAL FORM

An $n$-person game in normal form is a pair $\Gamma = (M, u)$, where $M := \prod_i M_i$ is a product of finite sets and $u = (u_1, \ldots, u_n)$ is an $n$-tuple of functions $u_i: M \rightarrow \mathbb{R}$. Here $M_i$ is the set of (pure) strategies of player $i$. 
and $u_i$ is his payoff function. For a strategy profile $x = (x_1, x_2, \ldots, x_n) \in \Delta_M := \prod_i \Delta(M_i)$ we define, as usual, the (expected) payoff function of player $i$ by

$$u_i(x) := \sum_{(k_1, k_2, \ldots, k_n) \in M} \prod_j (x_j)_{k_j} u_i(k_1, k_2, \ldots, k_n).$$

Furthermore, $(x_{-i} | y_i) \in \Delta_M$ is the strategy profile where player $i$ uses $y_i \in \Delta(M_i)$ and his opponents use the strategies in $x_{-i} \in \prod_{j \neq i} \Delta(M_j)$.

A game in strategic form is a pair $\langle P, v \rangle$, where $P = \prod_i P_i$ is a product of polytopes (not necessarily simplices) and $v = (v_1, \ldots, v_n)$ is an $n$-tuple of multiaffine functions $v_i : P \to \mathbb{R}$.

A solution for strategic form games is a map which assigns to each strategic form game $\Gamma = \langle P, v \rangle$ a collection of closed, nonempty subsets of $P$.

In order to describe the two methods of reducing a game, we call two strategies $x$ and $y$ of player $i$ payoff equivalent if $u_j(x_{-i} | x_i) = u_j(x_{-i} | y_i)$ for all $j$ and all $z_{-i} \in \prod_{j \neq i} \Delta(M_j)$.

The first way to reduce a normal form game is by eliminating pure strategies that are payoff equivalent with some possibly mixed strategy of the same player. The result of this reduction is again a normal form game.

The second way of reducing a game is based on the identification of arbitrary payoff-equivalent strategies. In order to show that this type of reduction may lead to a game in strategic (not necessarily normal) form we note that the payoff-equivalence relation on $\Delta(M_i)$ can be extended to an equivalence relation on $\mathbb{R}^M_i$. For $x_i, y_i \in \mathbb{R}^M_i$ and $z_{-i} \in \prod_{j \neq i} \Delta(M_j),$

$$u_j(z_{-i} | x_i) = u_j(z_{-i} | y_i) \iff u_j(z_{-i} | x_i - y_i) = 0 \iff x_i - y_i \in \text{Ker}[u_j(z_{-i} | \cdot)],$$

where $u_j(z_{-i} | \cdot)$ is the linear mapping $t_i \mapsto u_j(z_{-i} | t_i)$ on $\mathbb{R}^M_i$.

If we introduce the linear subspace

$$L_i := \bigcap_j \bigcap_{z_{-i}} \text{Ker}[u_j(z_{-i} | \cdot)]$$

of $\mathbb{R}^M_i$, the foregoing implies that the equivalence class containing $x_i \in \Delta(M_i)$ can be identified with the linear variety $\{x_i\} + L_i$. As is well known, the set $\mathbb{R}^M_i \mod L_i$ of all such linear varieties is a vector space isomorphic with the Euclidean space of dimension $|M_i| - \dim L_i$. So the set of equivalence classes containing at least one strategy of player $i$ can be identified with a subset of this Euclidean space. Since, for a strategy $x_i = \sum_k (x_i)_k e^k_i$ of player $i$ ($e^k_i$ is the vector in $\Delta(M_i)$ for which the $k$th
coordinate is equal to 1,
\[
\{x_i\} + L_i = \left( \sum_k (x_i)_k e_i^k \right) + L_i = \sum_k (x_i)_k \left[ e_i^k + L_i \right],
\]
this set of equivalence classes can be identified with a polytope, say \( P_i \), in a Euclidean space.

Finally, if we define for all \( i \)
\[
u_i(\{x_1\} + L_1, \ldots , (x_n) + L_n) := u_i(x_1, \ldots , x_n),
\]
we obtain a game of the form \( \langle P, \nu \rangle \), where \( P = \prod_i P_i \). Hence, the game that arises if for each player all payoff-equivalent strategies are identified is a game in strategic form. The following example shows that this reduction does not necessarily lead to a normal form game.

**Example 1.** For the \( 2 \times 4 \)-bimatrix game
\[
(A, B) = \begin{bmatrix}
(1, 1) & (-1, -1) & (2, -2) & (-2, 2) \\
(-1, -1) & (1, 1) & (-2, 2) & (2, -2)
\end{bmatrix}
\]
the subspace \( L_2 \) can be found as follows.

Two strategies \( x \) and \( y \) of player 2 are payoff equivalent if and only if
\[
\left\{ e_i Ax = e_i Ay \quad \text{for all } i \Leftrightarrow x - y \in L_2 := \left\{ \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} + \beta \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \mid \alpha, \beta \in \mathbb{R} \right\} \right. 
\]
Since two different pure strategies of player 2 are apparently not payoff equivalent, the strategy space \( P_2 \) can be identified with a quadrangle.

## 3. Reducing Games

In this section we describe a method introduced by Mertens (1987) (see also van Damme, 1996) of identifying two strategic form games. In the next section, this method plays an important role in defining the (weak) invariance of a solution for strategic form games. Furthermore we construct for each strategic form game a normal form game that can be identified with the original strategic form game.

First we will motivate Mertens' definition of reducing (strategic form) games. To that purpose we reconsider the second reduction method in the previous section. Corresponding to this way of reducing a game \( G \), we
consider the mapping \( f_i : \Delta(M_i) \rightarrow P_i \) defined by
\[
f_i(x_i) := \{ x_i \} + L_i.
\]
Then \( f_i \) is affine and surjective. Furthermore,
\[
u_i(x_1, \ldots, x_n) = v_i(\{ x_1 \} + L_1, \ldots, \{ x_n \} + L_n) = v_i(f_1(x_1), \ldots, f_n(x_n)).
\]
Hence, \( u_i = v_i \circ f \), where \( f := (f_1, \ldots, f_n) \). Apparently, the relation between a game and its reduced strategic form can be described by means of a mapping \( f \) as defined before. This leads to the following definition.

**Definition 1.** A strategic form game \( \Gamma' = \langle Q, w \rangle \) is a reduction of the strategic form game \( \Gamma = \langle P, v \rangle \) if there exists a mapping \( f = (f_1, \ldots, f_n) \), with for all \( i \)

1. \( f_i : P_i \rightarrow Q_i \) is affine and surjective
2. \( v_i = w_i \circ f \).

In this situation we write \( \Gamma \rightarrow_f \Gamma' \) and call \( f \) a reduction map from \( \Gamma \) onto \( \Gamma' \).

In Section 5 we will show that this definition restricted to normal form games is equivalent with the elimination of pure strategies (that are payoff equivalent with other strategies).

When all \( f_i \) are not only surjective but also injective, we have both \( \Gamma \rightarrow_f \Gamma' \) and \( \Gamma' \rightarrow_f \Gamma \). In that case we will call \( f \) an isomorphism between \( \Gamma \) and \( \Gamma' \) and say that \( \Gamma \) and \( \Gamma' \) are isomorphic.

**Remark 1.** If \( \Gamma = \langle M, u \rangle \) and \( \Gamma' = \langle L, v \rangle \) are two normal form games with \( \Gamma \rightarrow_f \Gamma' \), then the strategies \( x_i \) and \( y_i \) in \( M_i \) are payoff equivalent if and only if \( u_i(z_{-i} | x_i) = u_i(z_{-i} | y_i) \) for all \( j \) and all \( z_{-i} \in M_{-i} \). This is equivalent with \( v_i(f_i(z_k)_{k \neq i} | f_i(x_i)) = v_i(f_i(z_k)_{k \neq i} | f_i(y_i)) \) for all \( j \) and all \( z_{-i} \in M_{-i} \). Since \( f_i \) is surjective for all \( j \), this is equivalent with
\[
v_i(s_{-i} | f_i(x_i)) = v_i(s_{-i} | f_i(y_i)) \quad \text{for all } j \text{ and all } s_{-i} \in L_{-i}.
\]
This shows that \( f_i \) preserves payoff equivalence, i.e., \( x_i \) is payoff equivalent with \( y_i \), if and only if \( f_i(x_i) \) is payoff equivalent with \( f_i(y_i) \).

**Remark 2.** For a game in strategic form (best replies and equilibria can be defined in an obvious way. If \( B_f \) denotes the best reply correspondence for a strategic form game \( \Gamma \) and \( \Gamma \rightarrow_f \Gamma' \), then \( y \in B_f(x) \) if and only if \( f(y) \in B_{f'}(f(x)) \), as one easily verifies. Hence, \( x \in E(\Gamma) \) if and only if \( f(x) \in E(\Gamma') \).
Let $\Gamma = \langle P, v \rangle$ be a strategic form game. Corresponding to $\Gamma$ we will construct a normal form game $\Gamma_{\text{norm}} = \langle M, u \rangle$ such that $\Gamma$ is a reduction of $\Gamma_{\text{norm}}$. Since $P$ has a finite number of extreme points, there is a finite (index) set $M_i$ such that $\text{ext}(P_i) = \{ p_i^m \mid m \in M_i \}$. The linear map $\rho_i : \Delta(M_i) \to P_i$ defined by

$$\rho_i(x_i) := \sum_{m \in M_i} (x_i)_m p_i^m$$

is surjective. Next we consider the normal form game $\langle M, u \rangle$, with $M := \prod M_i$ and $u : M \to \mathbb{R}$ is the function defined by $u_i := u_i \circ \rho$. This game is called the normal form extension of $\Gamma$ and is denoted as $\Gamma_{\text{norm}}$. Furthermore, since for all $i$ and $x$

$$u_i(x) = \sum_{(k_1, \ldots, k_n) \in M} \prod_j (x_j)_{k_j} u_i(k_1, \ldots, k_n)$$

$$= \sum_{(k_1, \ldots, k_n) \in M} \prod_j (x_j)_{k_j} u_i(\rho_i(e_1^{k_1}), \ldots, \rho_i(e_n^{k_n}))$$

$$= \sum_{(k_1, \ldots, k_n) \in M} \prod_j (x_j)_{k_j} u_i(p_1^{k_1}, \ldots, p_n^{k_n})$$

$$= v_i \left( \sum_{k_1 \in M_1} (x_1)_{k_1} p_1^{k_1}, \ldots, \sum_{k_n \in M_n} (x_n)_{k_n} p_n^{k_n} \right)$$

$$= v_i(\rho_i(x_1), \ldots, \rho_i(x_n)) = (v_i \circ \rho)(x),$$

$\rho$ is a reduction map from $\Gamma_{\text{norm}}$ onto $\Gamma$.

Note that $\Gamma_{\text{norm}}$ is not uniquely determined because we only specified the number of elements of the index sets $M_i$, not the index sets themselves. However, it is clear that $\Gamma_{\text{norm}}$ is determined up to isomorphisms. Since any solution $\tau$ introduced in the literature satisfies

$$\tau(\Gamma) = \{ f(T) \mid T \in \tau(\Gamma) \}$$

if $\Gamma \to f \Gamma'$ and $f$ is an isomorphism, we may speak of the normal form extension of a game.

For later purposes we will show that the game $\Gamma_{\text{norm}}$ is a reduction of the game $\Gamma_{\text{norm}}$, if the game $\Gamma'$ is a reduction of the game $\Gamma$. Therefore we need the following result.

**Lemma 1.** If $f$ is a reduction map from $\Gamma = \langle P, v \rangle$ onto $\Gamma' = \langle Q, w \rangle$, then

$$\text{ext}(Q) \subset f_i(\text{ext}(P_i)).$$
Proof. Let \( q_i \in \text{ext}(Q_i) \). We will show that there exists a \( p_i \in \text{ext}(P_i) \) with \( f_i(p_i) = q_i \). Since \( f_i \) is surjective and affine, \( f_i^{-1}(q_i) \) is the nonempty intersection of \( P_i \) with a linear variety. Hence \( f_i^{-1}(q_i) \) is a polytope and we can choose an extreme point, say \( p_i \), of \( f_i^{-1}(q_i) \). In order to show that \( p_i \in \text{ext}(P_i) \), we suppose that \( p_i = \lambda x + (1 - \lambda)y \) with \( \lambda \in (0,1) \) and \( x, y \in P_i \). Then \( q_i = \lambda f_i(x) + (1 - \lambda)f_i(y) \) and \( f_i(x), f_i(y) \in Q_i \). Since \( q_i \) is an extreme point of \( Q_i \), this implies that \( q_i = f_i(x) = f_i(y) \). Then, however, \( x, y \in f_i^{-1}(q_i) \). Finally, the fact that \( p_i \) is an extreme point of \( f_i^{-1}(q_i) \) implies that \( x = y \), which completes the proof. \[ \blacksquare \]

**Lemma 2.** Let \( f \) be a reduction map from the strategic form game \( \Gamma = \langle P, v \rangle \) onto the strategic form game \( \Gamma' = \langle Q, w \rangle \). Then there exists a reduction map \( f^* \) from the normal form game \( \Gamma_{\text{norm}} = \langle L, u \rangle \) onto the normal form game \( \Gamma'_{\text{norm}} = \langle M, v' \rangle \) such that, for all \( i \), \( f_i \circ \rho_i = \rho_i \circ f_i^* \), where \( \Gamma_{\text{norm}} \to \rho \to \Gamma \) and \( \Gamma'_{\text{norm}} \to \rho' \to \Gamma' \).

Proof. As noticed before, \( \text{ext}(P) = \{ p_i^l \mid l \in L_i \} \) and \( \text{ext}(Q) = \{ q_i^m \mid m \in M_i \} \). In view of Lemma 1, we can find for any extreme point \( q_i^m \) of \( Q_i \) at least one extreme point, say \( p_i^l \), of \( P_i \) such that \( f_i(p_i^l) = q_i^m \). Therefore, without loss of generality, we may write \( L_i = M_i \cup K_i \) for some finite set \( K_i \).

We will construct a reduction map \( f^* \) from \( \Gamma_{\text{norm}} \) onto \( \Gamma'_{\text{norm}} \) such that the diagram

\[
\begin{array}{ccc}
\Delta_L & \xrightarrow{f^*} & \Delta_M \\
\downarrow{\rho} & & \downarrow{\rho'} \\
\rho & \downarrow{\rho} & Q \\
P & \xrightarrow{f} & Q \\
\end{array}
\]

commutes. Since \( \rho_i \) is surjective, we can find for all \( k \in K_i \) a strategy \( r_i^k \in \Delta(M_i) \) such that \( \rho_i(r_i^k) = f_i(p_i^k) \). Next we consider the linear mapping \( f_i^* : \Delta(L_i) \to \Delta(M_i) \) defined by

\[
 f_i^*(e_i^l) := \begin{cases} 
 e_i^l & \text{if } l \in M_i \\
 r_i^k & \text{if } l \in K_i 
\end{cases}
\]

First of all, note that \( f_i^* \) is surjective. Furthermore for an \( l \in M_i \)

\[
(f_i \circ \rho_i)(e_i^l) = f_i(p_i^l) = q_i^l = \rho_i(e_i^l) = \rho_i^*(f_i^*(e_i^l)) = (\rho_i^* \circ f_i^*)(e_i^l)
\]

and for an \( l \in K_i \)

\[
(f_i \circ \rho_i)(e_i^l) = f_i(p_i^l) = \rho_i(r_i^k) = \rho_i^*(f_i^*(e_i^l)) = (\rho_i^* \circ f_i^*)(e_i^l).
\]
Finally, since $u_i = v_i \circ \rho = w_i \circ f \circ \rho = w_i \circ \rho' \circ f^* = u_i' \circ f^*$, $f^*$ is a reduction map from $\Gamma_{\text{norm}}$ onto $\Gamma'_{\text{norm}}$ with the properties as mentioned in the theorem.

4. INVARIANT EXTENSIONS OF SOLUTIONS

Mertens and van Damme call a solution for strategic form games invariant if two requirements concerning the relation between the solution sets of a game and its reductions are satisfied. Since several solutions only satisfy the first of these requirements, we prefer to call such solutions weakly invariant. In this section we show that a (weakly) invariant solution for normal form games can be extended uniquely to a (weakly) invariant solution for strategic form games.

**Definition 2.** A solution $\tau$ is called weakly invariant if for all triplets $(\Gamma, \Gamma', f)$ with $\Gamma \rightarrow_f \Gamma'$,

$$\tau(\Gamma') = \{f(T) \mid T \in \tau(\Gamma)\}.$$ 

A weakly invariant solution $\tau$ is called invariant if, moreover, for all $S \in \tau(\Gamma')$,

$$f^{-1}(S) = \bigcup \{T \in \tau(\Gamma) \mid f(T) = S\}.$$ 

A solution for normal form games is a map which assigns to each normal form game $\Gamma = \langle M, u \rangle$ a collection of closed, nonempty subsets of $\Delta_M$. A normal form solution is called (weakly) invariant if the above holds for all triplets $(\Gamma, \Gamma', f)$ with $\Gamma$ and $\Gamma'$ normal form games.

We will show that a (weakly) invariant solution for normal form games can be extended uniquely to a (weakly) invariant solution for strategic form games. In the proof we need the following technical result.

**Lemma 3.** Let $\sigma$ be an invariant solution for normal form games, $\Gamma = \langle \Gamma_{\text{norm}}, v \rangle$ a strategic form game with $\Gamma_{\text{norm}} = \langle M, u \rangle$ and $\Gamma_{\text{norm}} \rightarrow_{\rho} \Gamma$. If $x \in \Delta_M$ satisfies $\rho(x) \in \rho(W)$ for some $W \in \sigma(\Gamma_{\text{norm}})$, then there exists a $V \in \sigma(\Gamma_{\text{norm}})$ containing $x$ such that $\rho(V) = \rho(W)$.

**Proof.** Since $\rho(x) \in \rho(W)$, there is a $y \in W$ such that $\rho(y) = \rho(x)$. Corresponding to $x$ we introduce a normal form game $\Gamma_{\text{norm}}[x]$ in such a way that $\Gamma_{\text{norm}}[x]$ is a reduction of $\Gamma_{\text{norm}}[x]$. In this new game each player gets an extra pure strategy, say $k_i$ for player $i$. In order to define the payoff function $u'_i$ for the new game, we consider the linear map $(\pi_i)_i : \Delta(M_i \cup$
\[(k, \cdot) \to \Delta(M_i) \text{ with} \]
\[
(p_i)_k(z) := \sum_{m \in M_i} (z)_m e^m_i + (z)_k x_i.
\]

We take \(u_i := u_i \circ \pi_i\) as the payoff function for player \(i\) in the new game. Clearly, \(\pi_i\) is a reduction map from the game \(\Gamma_{\text{norm}}[x]\) onto the game \(\Gamma_{\text{norm}}[y]\). Corresponding to \(y\) we construct, in a similar way, the normal form game \(\Gamma_{\text{norm}}[y]\) and a reduction map \(\pi_i\). By using Remark 1 with \(\pi_i\) and \(\pi_y\) in the role of \(f\), we find that, for all \(z\), \(u_i'(z) = u_i(\pi_i(z)) = u_i(\pi_y(z)) = u_i'(z)\). Hence, \(\Gamma_{\text{norm}}[x] = \Gamma_{\text{norm}}[y]\).

Now \(\pi_i(e_1^k, \ldots, e_n^k) = y \in W\). So, by the invariance of \(\sigma\), there exists a \(T \in \sigma(\Gamma_{\text{norm}}[y])\) containing \((e_1^k, \ldots, e_n^k)\) such that \(\pi_i(T) = W\).

Let \(V := \pi_i(T)\). Then \(x = \pi_i(e_1^k, \ldots, e_n^k) \in \pi_i(T) = V\) and \(V \in \sigma(\Gamma_{\text{norm}})\) in view of the weak invariance of \(\sigma\). The proof is complete if we can show that \(\rho(V) = \rho(W)\). For \(z_i \in \Delta(M_i \cup \{k\})\),

\[
\rho_i((\pi_i)_k(z)) = \rho_i \left( \sum_{m \in M_i} (z)_m e^m_i + (z)_k x_i \right)
\]

\[
= \sum_{m \in M_i} (z)_m \rho_i(e^m_i) + (z)_k \rho_i(x_i)
\]

\[
= \sum_{m \in M_i} (z)_m \rho_i(e^m_i) + (z)_k \rho_i(y_i)
\]

\[
= \rho_i \left( \sum_{m \in M_i} (z)_m e^m_i + (z)_k y_i \right) = \rho_i((\pi_i)_k(z)).
\]

Hence, \(\rho(V) = \rho(\pi_i(T)) = \rho(\pi_y(T)) = \rho(W)\).

**Theorem 1.** If \(\sigma\) is a weakly invariant solution for normal form games, then there exists a unique weakly invariant solution \(\tau\) for strategic form games such that \(\tau(\Gamma) = \sigma(\Gamma)\) for all normal form games \(\Gamma\). If \(\sigma\) is invariant, then \(\tau\) is invariant too.

**Proof.** Let for a strategic form game \(\Gamma\)

\[
\tau(\Gamma) := \{ \rho(T) \mid T \in \sigma(\Gamma_{\text{norm}}) \}
\]

with \(\Gamma_{\text{norm}} \to \sigma, \Gamma\). Note that \(\Gamma_{\text{norm}}\) was only determined up to isomorphisms. However, the assumption that \(\sigma(\Gamma_2) = \{ f(T) \mid T \in \sigma(\Gamma_2) \}\) if \(\Gamma_2 \to \Gamma_1\) is an isomorphism guarantees that our definition of \(\tau\) makes sense.
(a) In order to show that $\tau$ is weakly invariant, let $f$ be a reduction map from the strategic form game $\Gamma$ onto the strategic form game $\Gamma'$. Using Lemma 2, the weak invariance of $\sigma$ implies that

$$
\tau(\Gamma') = \{ \rho'(T) \mid T \in \sigma(\Gamma'_\text{norm}) \} = \{ \rho'(f^*(U)) \mid U \in \sigma(\Gamma_\text{norm}) \} = \{ f(\rho(U)) \mid U \in \sigma(\Gamma_\text{norm}) \} = \{ f(S) \mid S \in \tau(\Gamma) \}.
$$

So $\tau$ is weakly invariant.

(b) Furthermore, for a normal form game $\Gamma$, $\Gamma_\text{norm} = \Gamma$ and $\rho$ is the identity. So for such a game, $\tau(\Gamma) = \{ T \mid T \in \sigma(\Gamma) \} = \sigma(\Gamma)$.

(c) In order to show that $\tau$ is unique, let $\tau'$ be a weakly invariant solution for strategic form games such that $\tau'(\Gamma) = \sigma(\Gamma)$ for all normal form games $\Gamma$. If $\Gamma$ is a strategic form game, the fact that $\rho$ is a reduction map from $\Gamma_\text{norm}$ onto $\Gamma$, together with the weak invariance of $\tau'$, implies that

$$
\tau'(\Gamma) = \{ \rho(T) \mid T \in \tau'(\Gamma_\text{norm}) \} = \{ \rho(T) \mid T \in \sigma(\Gamma_\text{norm}) \} = \tau(\Gamma).
$$

(d) Finally, suppose that $\sigma$ is invariant and that $f$ is a reduction map from a strategic form game $\Gamma = \langle P, v \rangle$ onto a strategic form game $\Gamma' = \langle Q, w \rangle$. Let $S \in \tau(\Gamma')$ and let $p$ be an element of $P$ with $f(p) \in S$. We have to show that there exists a $T \in \tau(\Gamma)$ containing $p$ such that $f(T) = S$.

Write $\Gamma_\text{norm} = \langle L, u \rangle$. Since $\rho$ is surjective, there exists a $z \in \Delta_L$ such that $\rho(z) = p$. Then $\rho'(f^*(z)) = f(\rho(z)) = f(p) \in S$. By the definition of $\tau$ there is a $W \in \sigma(\Gamma'_\text{norm})$ with $\rho'(W) = S$. In view of Lemma 3 there is a solution set $V \in \sigma(\Gamma'_\text{norm})$ containing $f^*(z)$ such that $\rho'(V) = S$. By the invariance of $\sigma$, there exists a solution set $U \in \sigma(\Gamma_\text{norm})$ containing $z$ such that $f^*(U) = V$. For the set $T := \rho(U)$ we have $T \in \tau(\Gamma)$ and $p = \rho(z) \in \rho(U) = T$. Finally,

$$
f(T) = f(\rho(U)) = \rho'(f^*(U)) = \rho'(V) = S.
$$

In view of this result, it is—in the context of invariance—sufficient to consider solutions for normal form games only. In the next section we will show that the invariance of such solutions can be described by means of so-called extension sets.
5. THE INVARIANCE OF NORMAL FORM SOLUTIONS

In this section we will show that a normal form game is the reduction of another one if and only if one of the games arises from the other one by adding convex combinations of pure strategies as new pure strategies. To that purpose, we include these combinations, which can be represented by probability vectors, in a so-called extension set. This extension set is used to construct a reduction map. To be precise, let \( \Gamma = \langle M, u \rangle \) be an \( n \)-person game in normal form.

**Definition 3.** For each player \( i \), let \( P_i = \{ p_i^k \in \Delta(M_j) \mid k \in K_i \} \) be a finite set of strategies. We call the set of vectors \( P := \cup P_i \) an extension set for \( \Gamma \). For such a set \( P \) we introduce the \( P \)-extended game \( \Gamma_p = \langle M', u' \rangle \), where \( M' := M_i \cup K_i \) is the disjoint union of the sets \( M_i \) and \( K_i \). In order to define the payoff functions of this game, we consider for all \( i \) the linear map \( \pi_i: \Delta(L_i) \to \Delta(M_i) \) with

\[
\pi_i(e_i^l) := \begin{cases} e_i^l & \text{if } l \in M_i, \\ p_i^l & \text{if } l \in K_i. \end{cases}
\]

The payoff function \( u'_i: \Delta_L \to \mathbb{R} \) is then defined by \( u'_i := u_i \circ \pi \), where \( \pi := (\pi_1, \ldots, \pi_n) \). The projection \( \pi: \Delta_L \to \Delta_M \) defined in this way is sometimes denoted by \( \pi_P \) to avoid a possible misinterpretation.

Obviously, \( \Gamma_\phi = \Gamma \). Furthermore, \( \Gamma_p \) can be considered as an extension of \( \Gamma \) since the game that results when each player \( i \) in \( \Gamma_p \) is restricted to his pure strategies in \( M_i \) is exactly \( \Gamma \).

**Example 2.** For the \( 2 \times 2 \)-bimatrix game \( \Gamma \) given by

\[
\begin{bmatrix}
(1,2) & (0,0) \\
(0,0) & (2,1)
\end{bmatrix},
\]

we choose \( P_1 = \{(\frac{1}{2}, \frac{2}{3})\}, P_2 = \phi, \pi_1: \Delta_3 \to \Delta_2 \) as the map defined as

\[
\pi_1(e_i) = \begin{cases} e_i & \text{if } i = 1, 2 \\ \left(\frac{1}{3}, \frac{2}{3}\right) & \text{if } i = 3 \end{cases}
\]

and \( \pi_2: \Delta_2 \to \Delta_2 \) as the identity map.
Obviously, $\Gamma_p$ is the $3 \times 2$-bimatrix game given by
\[
\begin{pmatrix}
(1,2) & (0,0) \\
(0,0) & (2,1) \\
\left(\frac{1}{3},\frac{2}{3}\right) & \left(\frac{4}{3},\frac{2}{3}\right)
\end{pmatrix}.
\]

Clearly, $\Gamma_p \rightarrow \Gamma$. In the following theorem we show that any reduction map for normal form games can be obtained in this way.

**Theorem 2.** If $f$ is a reduction map from the normal form game $\Gamma' = \langle L, v \rangle$ onto the normal form game $\Gamma = \langle M, u \rangle$, then there is an extension set $P$ for the game $\Gamma$ such that $f = \pi_p$ and $\Gamma' = \Gamma_p$.

**Proof.** As in the proof of Lemma 2 one can show that, for any $i$, $L_i = M_i \cup K_i$ for some finite set $K_i$, and that $f_i(e_i^m) = e_i^m$ for each $m \in M_i$.

Now we consider the set $P := \{f_i(e_i^k) \mid k \in K_i\}$. Then $P := \bigcup_i P_i$ is an extension set for the game $\Gamma$. Moreover, since for all $i$,
\[
\pi_i(e_i^l) = \begin{cases} e_i^l = f_i(e_i^l) & \text{if } l \in M_i \\ f_i(e_i^l) & \text{if } l \in K_i, \end{cases}
\]

it follows that $\pi_p = f$. Consequently, $\Gamma' = \Gamma_p$. 

**Corollary 1.** A solution $\sigma$ for normal form games is weakly invariant if and only if for any game $\Gamma$ and any extension set $P$ for $\Gamma$,
\[
\sigma(\Gamma) = \{\pi_p(T) \mid T \in \sigma(\Gamma_p)\}.
\]

A weakly invariant solution $\sigma$ is invariant if and only if for any game $\Gamma$ and any extension set $P$ for $\Gamma$,
\[
\pi_p^{-1}(S) = \bigcup \{T \in \sigma(\Gamma_p) \mid \pi_p(T) = S\}
\]

for all $S \in \sigma(\Gamma)$.

**Proof.** (a) Suppose that $\sigma$ is a weakly invariant solution for normal form games. If $P$ is an extension set for a normal form game $\Gamma$, then $\Gamma_p \rightarrow \Gamma_p$. Hence,
\[
\sigma(\Gamma) = \{\pi_p(T) \mid T \in \sigma(\Gamma_p)\}
\]
by the weak invariance of $\sigma$.

(b) Suppose, conversely, that for any game $\Gamma$ and any extension set $P$ for $\Gamma$,
\[
\sigma(\Gamma) = \{\pi_p(T) \mid T \in \sigma(\Gamma_p)\}.
\]
Let $f$ be a reduction map from a normal form game $\Gamma$ onto a normal form game $\Gamma'$. Then by Theorem 2 there is an extension set $P$ for the game $\Gamma$ such that $f = \pi_p$ and $\Gamma' = \Gamma'_p$. Hence $\sigma$ is weakly invariant because

$$\sigma(\Gamma) = \{\pi_p(T) \mid T \in \sigma(\Gamma'_p)\} = \{f(T) \mid T \in \sigma(\Gamma')\}.$$ 

In a similar way a proof can be given for the invariance of a solution. ■

REFERENCES

