THE BEST BUSINESS CYCLE INDICATOR: A MEDIUM n APPROACH WITH APPLICATION TO EU COUNTRIES

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Abstract

Combining series with the aim to obtain an indicator for business cycle analyses is an important issue for policy makers for instance. In this area, econometric techniques usually rely on small systems (VAR or VECM) or at the other extreme on models for a very large number of series (factor models). In this paper we propose tools for medium n, situations that are likely to be the most frequent in empirical works. Studying jointly the 27 EU countries is an example. We show how to build from individual ARMA models, a simple indicator that is coherent with co-movements observed in a multivariate model but that cannot be estimated due to lack of degrees of freedom.

JEL: C32
Keywords: Co-movements, ARMA, EU business cycle indicator, aggregation, final equations.

This is a first draft version

1 Introduction

We consider two definitions of the business cycle fluctuations. On the one hand there is the Classical Cycle approach (BCcc hereafter, where "BC" and "cc" stand for respectively business cycle and classical cycle), namely the analysis of business cycle features (phases, turning points,...) extracted in the level of variables measuring the economic activity. On the other hand there is

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the Growth Cycle analysis (BC$_{gc}$ hereafter), hence the business cycle facts derived in economic series after an "appropriate" extraction of the trend component. Obviously, the quotation marks around appropriate refer to the well know observation that different smoothing techniques such as the deterministic trend, the Hodrick-Prescott (HP), the Baxter-King or the Beveridge-Nelson (BN) filter to name a few, deliver alternative detrended series and therefore possibly different BC$_{gc}$ facts (see e.g. Canova (1998)).

Our concern in this paper is not to further investigate this important issue. Instead we focus on the choice of an aggregate for the extraction of business cycle features, either for BC$_{cc}$ or BC$_{gc}$. We develop a strategy aiming to obtain our "best" linear combination of individual components. This humble terminology simplifies in a provocative manner the aim of this paper: to determine for the gross domestic product of EU state members a simple aggregate in such a way that the number of co-movements among these countries is maximized. Roughly speaking, we propose for example to determine turning points on a global variable that is compatible with a underlying statistical multivariate model for possibly 25 countries. However, and this is the novelty of our approach, we do so without estimating a multivariate model. Indeed, all computations only require the identification and estimation of individual ARIMA models, possibly on different period of observations for different series.

The paper is organized as follows. The second section states the notations and summarizes different ideas regarding the detection of business cycle fluctuations in VECMs with additional common cyclical feature relationships. These tools can only be applied for a small number of variables. Section 3 gives the main findings about the final equations of the VECM with additional cyclical co-movements. Section 4 proposes a way to obtain coherent aggregates from individual processes. Section 5 applies this methodology to the identification of an EU aggregate that is obtained for different data span.

## 2 BC$_{cc}$ and BC$_{gc}$ in VECMs with a small $n$

Let us denote $\Phi(L)Z_t = \Theta D_t + \varepsilon_t$ the $n$-dimensional vector autoregressive model of order $p$, i.e. the VAR($p$), for the $I(1)$ variables $Z_t = (Z_{1t}, \ldots, Z_{nt})'$, for fixed values of $Z_{-p+1}, \ldots, Z_0$ and with $\Phi(L) = I_n - \Phi_1 L - \ldots - \Phi_p L^p$. $D_t$ is a vector of deterministic terms such as an intercept or seasonal dummies, and the disturbances $\varepsilon_t$ are $NIID(0,-\lambda)$. Let us further assume that $\text{rank} [\Phi(1)] = r$, $0 < r < n$, so that $\Phi(1) = I_n - \Phi_1 - \ldots - \Phi_p$ can be expressed as $\Phi(1) = -\alpha \beta'$, with $\alpha$ and $\beta$ both $(n \times r)$ matrices of full column rank $r$ and that the characteristic equation $|\Phi(z)| = 0$ has
n – r roots equal to 1 and all other roots outside the unit circle. The multivariate process \( Z_t \) is then cointegrated of order (1,1). The columns of \( \beta \) span the space of cointegrating vectors, and the elements of \( \alpha \) are the corresponding adjustment coefficients or factor loadings. Decomposing the matrix lag polynomial \( \Phi(L) = \Phi(1)L + \Gamma(L)(1 - L) \), and defining \( \Delta = (1 - L) \), we obtain the vector error correction model

\[
\Delta Z_t = \Theta D_t + \alpha \beta' Z_{t-1} + \sum_{j=1}^{p-1} \Gamma_j \Delta Z_{t-j} + \varepsilon_t, \quad t = 1, \ldots, T, \tag{1}
\]

where \( \Gamma_0 = I_n \), \( \Gamma_j = -\sum_{k=j+1}^{p} \Phi_k \) (\( j = 1, \ldots, p - 1 \)).

### 2.1 Aggregates for BC\(_{cc}\): PCA and other RRR

As far as a combination of the series \( \Delta Z_t \) is required, for example to build a coincident indicator of the economic activity, a simple principal component analysis (PCA) can be carried out. It is well known that PCA consists in collecting the eigenvector associated with the largest eigenvalue of the correlation matrix of \( \Delta Z_t \) to obtain the combination \( \nu' Z_t \) on which BC\(_{cc}\) are determined. However a PCA only uses the variables \( \Delta Z_t \), i.e. the lhs of (1). Alternative techniques also use the information from a dynamic parametric model. In the sequel we only consider some approaches whose restrictions can be derived from the VECM (1). In particular, these techniques involve additional restrictions associated with short-run co-movements between the series in first differences.

Let us add to the VECM (1) restrictions coming from common dynamic features. In this framework, serial correlation common feature (SCCF hereafter, see Engle and Kozicki 1993) holds for the VECM in (1), if there exists a \((n \times s)\) matrix \( \delta \), whose columns span the cofeature space, such that

\[
\delta' (\Delta Z_t - \Theta D_t) = \delta' \varepsilon_t, \quad t = 1, \ldots, T,
\]

is a s-dimensional zero mean vector innovation process with respect to the information available at time \( t \). Consequently, SCCF arises if there exists a matrix \( \delta \) such that the conditions \( \delta' \Gamma_j = 0_{(s \times n)} \), \( j = 1 \ldots p - 1 \) and \( \delta' \Phi(1) = -\delta' \alpha \beta' = 0_{(s \times n)} \) are jointly satisfied. To form a linear combination that is the most correlated with the past and therefore the "most dynamic relationship", Issler and Vahid (2006) look at BC\(_{cc}\) on \( \delta'_{\perp} Z_t \) where \( \delta'_{\perp} \) is the orthogonal space to \( \delta \). Indeed, if \( \delta \) gives the vectors that produce the combination of \( \Delta Z_t \) which is orthogonal to the past (i.e. zero eigenvalues

\(^1\)For instance we do not consider Harvey’s state space approach.
in a canonical correlation problem), \( \delta_\perp \) has the opposite meaning. Remark that Hecq (2005) has illustrated that for the US business cycle coincident indicator, any linear combination of the four series (PCA, SCCF or a simple average) roughly gives the same historical peaks and throughs for the \( BC_{cc} \) definition.

Alternatively and to jointly determine a coincident and a leading business cycle indicator, Cubadda (2007) searches for polynomial SCCF of order one (PSCCF(1) hereafter, see Cubadda and Hecq (2001)) relationships in a VECM (1), that is to say a matrix \( \delta_0 \) such that under the null hypothesis that PSCCF holds, the conditions \( \delta_0^j \Gamma_j = 0_{(s \times n)} \), \( j = 2 \ldots p - 1 \) and \( \delta_0' \Phi(1) = -\delta_0' \alpha_0 \beta' = 0_{(s \times n)} \) are jointly satisfied such that there exists an \( n \times s \) polynomial matrix with

\[
\delta'(L) \Delta Z_t \equiv (\delta_0 + \delta_1 L)' \Delta Z_t = \delta_0 \Theta D_t + \delta_0 \varepsilon_t.
\]

Then an optimal coincident indicator, in which the \( BC_{cc} \) can be determined, is given by \( \delta_0 \varepsilon_t \).\(^2\)

### 2.2 Aggregates for \( BC_{gc} \)

The VECM with both long and short-run restrictions also gives a full description of the common trends and cycles of \( Z_t \), useful for a \( BC_{gc} \) study. Indeed, from the Wold representation of the stationary process \( \Delta Z_t \) and focusing on the Beveridge-Nelson decomposition (ignoring deterministic terms for the simplicity of notations here) we have

\[
\Delta Z_t = C(L) \varepsilon_t, \\
= C(1) \varepsilon_t + \Delta C^*(L) \varepsilon_t,
\]

with \( C(L) = I_n + \sum_{i=1}^{\infty} C_i L^i \) and \( \sum_{j=1}^{\infty} j |C_j| < \infty \) and \( C_i^* = -\sum_{j>i}^{\infty} C_j \) for all \( i \). Integrate both sides of (2) we obtain

\[
Z_t = C(1) \sum_{j=1}^{t} \varepsilon_j + C^*(L) \varepsilon_t, \\
= Trends + Cycles.
\]

These trends and cycles will be common to the series depending on the rank of the matrices \( C(1) \) and \( C^*(L) \). Under cointegration \( \text{rank}(C(1)) = n - r \) and we know that the \( n - r \) common stochastic

\[^2\text{Both Cubadda (2007) and Issler and Vahid (2006) show how to combine the vectors of the matrix } \delta_0 \text{ and } \delta_\perp \text{ when the rank of these matrices is larger than 1.}\]
trends $\alpha \sum_{j=1}^{t} \varepsilon_j$ are annihilated by $\beta$ because $\beta' C(1) = 0$. Similarly, under SCCF for instance, $C^*(L)$ is of reduced rank $n - s$ and these $n - s$ common cycles are such that $\delta' C^*(L) = 0$ (see inter alia Issler and Vahid 2001). This also illustrates, that under SCCF, BC$_{gc}$ may well be obtained on $\beta' Z_t = \beta' Cycles$ (see inter alia Gonzalo and Granger (1995), Proietti (1997), Hecq, Palm and Urbain (2000)). See also Cubadda (2007) for BN cycles under PSCCF.

2.3 Some practical problems

However, and whatever the definition of business cycle fluctuations chosen by a practitioner, the analysis derived from these multivariate approaches is altered by at least four drawbacks. The first problem is the well known dimensionality issue. Indeed the maximum number of variables that can be included in such a multivariate modelling is relatively small. If it is not the case, namely when one must work with a large data set, alternative methods such as the dynamic factor model may be more appropriate. However this latter approach is feeded by an asymptotics on $n$, something we do not have for many interesting cases. As an example, we consider 25 countries in this paper, thus too many series for usual VAR-VECM models but too few variables to adequately rely on the assumptions of factor models. A second problem of the multivariate approach is its sensitivity to various misspecifications. For instance the presence of ARCH may affect both a cointegration and a common feature analysis, the incorrect choice of the dynamics has important consequences, the use of seasonally adjusted data instead of raw variables destroys the properties of common feature tests statistics. The econometric literature is full of Monte Carlo study and empirical analyses commenting these issues that we shall therefore not further develop. A third problem comes from the fact that VAR-VECM analyses suppose the same time span for every series, that is to say a nice set of time balanced variables. Finally, even if this point is often neglected, we must confess that it is not always obvious to choose the variables that enter in an empirical study. One might for instance be tempted to work with the largest economies first.

The approach developed in this paper tries to minimize these drawbacks. The trick is to use an univariate framework which is compatible with the underlying multivariate VAR-VECM. Consequently we keep advantage of both the economic content of a multivariate modeling and the simplicity and the robustness of individual ARIMA models. The final equation framework allows to bridge that gap.
3 The final equations representation

Our goal is to aggregate series in a simple manner. The issue is to consider the variables that share the maximum number of short-run co-movements but whose detection is performed using univariate ARIMA models. Let us assume that the underlying DGP for \( n \) time series \( Z_t = (Z_{1t}, \ldots, Z_{nt})' \) is a potentially large, possibly non-stationary, \( \text{VAR}(p) \),

\[
\Phi(L)Z_t = \Theta D_t + \varepsilon_t, \quad t = 1 \ldots T, \tag{3}
\]

where \( \Phi(L) \), \( \Theta D_t \) and \( \varepsilon_t \) have already been defined. Let us then consider the implied structure of that \( \text{VAR} \) or \( \text{VARMA} \), called the final equations (FE hereafter) by Zellner and Palm (1974, 1975), Palm and Zellner (1980) by multiplying both sides (3) by the adjoint of \( \Phi(L) \) using \( \Phi(L)^\text{adj} = \det[\Phi(L)]\Phi^{-1}(L) \) such that

\[
\det[\Phi(L)]Z_t = \tilde{\Theta} D_t + \Phi(L)^\text{adj} \varepsilon_t, \tag{4}
\]

where the determinant \( \det[\Phi(L)] \) is a scalar finite polynomial in \( L \) and \( \tilde{\Theta} D_t = \Phi(L)^\text{adj} \Theta D_t \). The FE representation (4) shows that each series is a finite order \( \text{ARMA}(p^*, q^*) \), with the same lag structure and the same coefficients for the autoregressive part for every series, although the system was a finite order \( \text{VAR}(p) \). For instance for the \( i \)th element of \( Z_t \) we have \( \det[\Phi(L)]Z_{it} = \tilde{\Theta}_i D_t + \sum_{j=1}^N \Phi_{ij}(L)^\text{adj} \varepsilon_{jt} = \tilde{\Theta}_i D_t + \theta_i(L)u_{it}, \quad i = 1 \ldots n \).

A small order \( \text{VAR}(p) \) with few series already generates univariate ARMA model with large \( p^* \) and \( q^* \), an observation that is rejected empirically. Indeed, an \( n \) dimensional \( \text{VAR}(p) \) would imply at most individual \( \text{ARMA}(np, (n-1)p) \) processes.\(^3\) Cubadda, Hecq and Palm (2007a,b) show that these maximum numbers can be considerably reduced if we highlight the presence of common cycles (e.g. SCCF, PSCCF) in the joint dynamics of these \( n \) individual series. The next example illustrates this point for a non-stationary \( \text{VAR}(2) \) with three cointegrated I(1) series.

**Example 1** For a \( \text{VAR}(2) \) \( \Phi(L)Z_t = (I - \Phi_1 L - \Phi_2 L^2)Z_t = \varepsilon_t \) with the numerical values

\[
\Phi(L) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} - \begin{bmatrix}
0.625 & 0.1875 & -0.375 \\
0.75 & 0.625 & 0.75 \\
-0.75 & 0.375 & 0.25 \\
\end{bmatrix} L - \begin{bmatrix}
0.125 & 0.0625 & 0.125 \\
-0.25 & -0.125 & -0.25 \\
0.25 & 0.125 & 0.25 \\
\end{bmatrix} L^2,
\]

\(^3\)This well know result is simply due to the fact that \( \det[\Phi(L)] \) contains by construction up to \( L^{np} \) terms and the adjoint matrix is a collection \( (n-1) \times (n-1) \) cofactor matrices, each of the matrix elements can contain the terms 1, \( L, \ldots L^p \).
the determinant $\det[\Phi(L)] = L^3 - 0.25L^2 - 1.5L - 0.25L^4 + 1.0$ has roots \{1, 1, -1.236, 3.236\}. These fancy numbers for $\Phi(L)$ becomes clearer if we write the VAR in its VECM form $\Delta Z_t = \alpha \beta' Z_{t-1} - \Phi_2 \Delta Z_{t-1} + \varepsilon_t$ with additional SCCF restrictions, hence

$$\Delta Z_t = \begin{bmatrix} -0.25 \\ 0.5 \\ -0.5 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} Z_{t-1} - \begin{bmatrix} 0.125 \\ -0.25 \\ 0.25 \end{bmatrix} \begin{bmatrix} 1 & 0.5 & 1 \end{bmatrix} \Delta Z_{t-1} + \varepsilon_t.$$  

We see that there exists a SCCF matrix

$$\delta' = \begin{bmatrix} 1 & 0 & -0.5 \\ 0 & 1 & 1 \end{bmatrix},$$

that annihilates the dynamics and consequently $\alpha$ and $\Phi_2$ such that $\delta' \Delta Z_t = \delta' \varepsilon_t$. The adjoint of $\Phi(L)$ has also one unit root per variable that is cancelled out with one of the unit root in the autoregressive component. Consequently the three univariate implied processes (i.e. the FE) are at most ARIMA(2,1,2) models and not ARIMA(4,1,3) models such as in a VAR without short-run reduced rank restrictions.

Table 1, taken from Cubadda, Hecq and Palm (2007b), generalizes the previous example. SCCF and PSCCF have already been defined in the previous section. WF stands for the weak form common feature representation (Hecq, Palm and Urbain (2006)) a modeling in which the loading and the short-run matrices of the VECM do not share the same left null space, and thus leads to

$$\delta' (\Delta Z_t - \alpha \beta' Z_{t-1}) = \delta' \alpha + \delta' \varepsilon_t.$$  

Table 1 gives an alternative explanation by the presence of co-movements, to the contradiction between the FE representation and the parsimonious univariate ARIMA models one observe in empirical investigation.

These results are also coherent with practices in both dynamic factor models and panel data analyses. Indeed and assume that the world is approximately generated by a huge VAR (or VARMA), observing parsimonious univariate ARMA models is compatible with the statement that there exists only few long-run relationships (otherwise orders would be higher because $r$ is added) and a lot of short-run co-movements $s$ (and so only a few common factors). This is basically one of the working assumption of Forni et al. (2000).

For panel data analyses and in particular for the new generation of panel unit root tests, our
Table 1: Maximum ARIMA orders of univariate non-stationary series generated by a CI(1,1) VAR(p) with cofeature restrictions

<table>
<thead>
<tr>
<th>Models</th>
<th>AR order</th>
<th>MA order</th>
</tr>
</thead>
<tbody>
<tr>
<td>CI(1,1)-VAR(p)</td>
<td>(n(p - 1) + r)</td>
<td>((n - 1)(p - 1) + r)</td>
</tr>
<tr>
<td>SCCF</td>
<td>((n - s)(p - 1) + r)</td>
<td>((n - s)(p - 1) + r)</td>
</tr>
<tr>
<td>PSCCF(1)</td>
<td>((n - s)(p - 1) + r + s)</td>
<td>((n - s)(p - 1) + r + s - 1)</td>
</tr>
<tr>
<td>WF</td>
<td>((n - s)(p - 1) + r)</td>
<td>((n - s)(p - 1) + r)</td>
</tr>
</tbody>
</table>

interpretation favors test statistics with (i) an homogeneous slope, (ii) some heterogeneity in the dynamics, (iii) and cross-sectional correlation (see Cubadda, Hecq and Palm (2007a)). But we show using the FE implied by the VECM that the presence of additional hidden cointegrating vectors is already taken into account in the degree of the ARMA models and consequently does not deserve a particular treatment.

4 Coherent Aggregation, selection and testing

Our analysis implies that for obtaining a coherent aggregate, the series that must be considered are those with similar dynamic properties. These might be the FE of an underlying multivariate process. Let us also start to assume that we have found such variables before we propose a strategy to get them.

4.1 Aggregating series

Under the null hypothesis that individual ARMA models are indeed the implied processes from a VAR(p) with short-run co-movements, Cubadda, Hecq and Palm (2007a) give evidence using a Monte Carlo experiment that the best strategy is to simply run the parsimonious empirical ARIMA model for the average \(\bar{\Delta}Z_t = \frac{1}{n} \sum_{i=1}^{n} \Delta Z_{it}\).\(^4\) This method of estimation of the common parameter is trivial to implement, and gives better results in terms of bias and RMSE than the mean group of individual series for instance. The intuitive reason is obvious: under the null hypothesis, averaging series imposes the common autoregressive component and often reduces the effect of the MA contribution because the adjoint has a factor representation (see Cubadda, Hecq and Palm (2007b)).

\(^4\)Alternatively a PCA can be performed. A weighted average might be appropriate as well.
This simple observation is valid if we aggregate series that are the implied equations and/or if we include some ARMA with the same autoregressive root. In most cases it is not easy to determine whether the coefficients of ARMA models are close to each other or not. In Cubadda, Hecq and Palm (2007b) we have explained that it is difficult to compare the unconstrained variance covariance matrix (as well as the autocovariances) with the constrained matrices in which we would impose a common autoregressive part. Also the full maximum likelihood approach by Palm and Zellner (1980) is rather tedious to implement in large systems. This is the reason why we look at a very simple strategy we summarize in the following steps:

- (i) We first test for unit roots in each series individually using for instance an ADF Student—t test. Then we take the first differences of the series for which we do not reject the null of a unit root.

- (ii) We identify the $n$ univariate AR(I)MA models. This can be done using EACF and EPACF as well as information criteria. In the sequel we use a SBC in the Hannan-Rissanen approach (see for instance Lütkepohl and Krätzig (2004)). This approach consists in estimating a large AR for the series and to take the residuals as a proxy of the MA component. Then a set of ARMA$(p,q)$ is computed. SBC is used to select the most parsimonious model although the sensitivity of the results to other criteria is worth considering. There are pros and cons of this approach but it has the advantage to not select an ARMA model subject to common factor, a model that is often detected if one use information criteria after fitting all possible ARMA models directly on the series. Moreover, mechanically determining ARMA models with information criteria avoids some subjective interpretation of the results in this kind of analysis. For instance we could be tempted to see more commonality for the EU core countries.

- (iii) We combine series by simple average or by principal component. Taking the average is easier when series do not have the same period of observation. For instance for two variables $\Delta Z_{it} \ t = 1 \ldots T$ and $\Delta Z_{jt} \ t = t^* \ldots T$ with $t^* > 1$, we compute an aggregate as follows:

$$
\Delta Z_t = \begin{cases} 
\Delta Z_{it}, & t < t^* \\
\frac{1}{2}(\Delta Z_{it} + \Delta Z_{jt}), & t \geq t^*
\end{cases}
$$
We can also imagine to combine series with different weights such as in

$$
\Delta Z_t = \begin{cases} 
\frac{1}{\omega_i} \Delta Z_{it}, t < t^*, \\
\frac{1}{\omega_i} \Delta Z_{it} + \frac{1}{\omega_j} \Delta Z_{jt}, t \geq t^*, \\
\frac{1}{\omega_i} + \frac{1}{\omega_j} = 1 
\end{cases}
$$

but the choice of these weights is not trivial: What indicator must we consider? Are the weights constant through time?

- (iv) After having estimated $\Delta Z_t$, we look at the parsimonious ARMA($\bar{p}, \bar{q}$) model as well as the $p - value$ from a test for the null of no autocorrelation. For that test we favor an LM[1 to $h$] to a Box-Piece type whose $\chi^2_{(h-p-q)}$ distribution changes with the estimated ARMA($p, q$) model.

- (v) Two series might be the FE of the same VAR if and only if $\bar{p} \leq \max(p_i, p_j)$ and $\bar{q} \leq \max(q_i, q_j)$ and if we do not reject the null of no autocorrelation in the ARMA($\bar{p}, \bar{q}$) estimated model.

4.2 Selecting variables

The whole method assumes that we must know the series to aggregate. This is not obviously the case in practice and techniques for selecting variables are proposed in this subsection. We propose four different metrics whose properties must still be evaluated in finite and large samples.

4.2.1 Metric 1: $M_1$

For the first measure, we compute bivariate Euclidean distances for every pair of AR($\infty$) coefficients (see Corduas and Piccolo (2008) and references therein) using

$$
d_{(i,j)}^{M_1} = \sqrt{\sum_{k=1}^{\infty} (\pi_{i,k} - \pi_{j,k})^2},
$$

where for two series $\Delta Z_{it}$, $\Delta Z_{jt}$ of the $n$ vector $\Delta Z_t$, $i, j = 1 \ldots n$, $\pi_{i,k}$ and $\pi_{j,k}$ are the scalar parameters of the AR($\infty$). However and contrarily to Corduas and Piccolo (2008) who assume independence of innovations, we cannot use a test statistics for the null hypothesis that a distance is equal to zero. Indeed two series are not independent by definition if they are implied by the same VAR.
Consequently we only collect these distances in an $n \times n$ matrix $D_{M1}^n$ and proceed as follows: We first find the minimum value of the distance in each columns of $D_{M1}^n$ and we aggregate these two series. Then we compute again that matrix using the new aggregate, i.e. a new $(n-1) \times (n-1)$ matrix $D_{M1}^{n-1}$. Two cases are possible: if there exist a smaller value of the distance in $D_{M1}^{n-1}$ for the bivariate case than in $D_{M1}^n$, we consider in addition to the first aggregate, this new combination; if the combination of three variables is smaller, we aggregate these three series and go ahead with the distances in a matrix $D_{M1}^{n-2}$. We continue until the last variable.

4.2.2 Metric 2: $M_2$

It is interesting to look at the previous metric, and this is what we do in this paper. However one quickly notices that the conditions on AR($\infty$) coefficients are not necessary but only sufficient for our analysis. Indeed the FE results, summarized in Section 3, only imply a common finite AR component, not a similarity of the coefficients of the overall ARMA model. The FE representation means that for two series we have

$$a(L)\Delta Z_{it} = \theta_i(L)u_{it},$$

$$a(L)\Delta Z_{jlt} = \theta_j(L)u_{jlt},$$

and consequently possibly different AR($\infty$) representations: $\pi_i(L) = a(L)/\theta_i(L)$ and $\pi_j(L) = a(L)/\theta_j(L)$. We therefore also propose to compute the Euclidean distance but on the parameters of the AR part only, that is

$$d_{M_2}^{i,j} = \sqrt{\sum_{k=1}^{p} (a_{i,k} - a_{j,k})^2}.$$

This measure has however important practical drawbacks. First, it is not easy to identify ARMA model for every series. In the empirical section we have used a SBC but this not prevent us from an incorrect identification of the ARMA orders. Alternatively, we have tried to estimate the same ARMA($p,q$) models on all variables. We have taken the orders $p = \max(p_i)$ and $q = \max(q_i)$, the degrees obtained as the maximum orders for the AR and the MA part. Unfortunately this procedure often overestimate the ARMA part. This leads to common factors that increase the value of the AR parameters and yields non invertible MA roots. Reducing the MA degree produces a radically different ranking of the series, but common factors are still present.

On the other hand, we might use the ARMA model identified on each series and compute the Euclidean distance on the AR part only. When comparing an AR(4) with an AR(3), we
then put a zero for the fourth coefficient of the latter variable. However this leads to impose a zero distance for all white noise process as well as between a white noise and an ARMA(0,2) for instance. Consequently and before we obtain more convincing results we prefer to work with the sufficient measurement 1.

4.2.3 Metric 3: $M_3$

Alternatively we can use the ML test in Palm and Zellner (1980) for all pairs and take the $p-values$ as a measure of the distances such that

$$d_{(i,j)}^{M_3} = p_{val_{i,j}}$$

The advantage is that this approach allows to formally test whether the countries have a same AR part. The drawback is that the distribution of the test must be adapted because we should consider the distribution of the minima. Moreover, only series with the same period can be considered.

4.2.4 Metric 4: $M_4$


In this version of the paper we only consider metrics 1.

5 Empirical Analysis

Let us apply the approach proposed in this paper and try to detect co-movements within a set of 25 gross domestic product, namely the 27 EU state members minus Romania and Bulgaria for which we do not have the relevant data. Quarterly seasonally adjusted series have been downloaded from Eurostat short-term indicators database (release October 2007).

Table 4 in appendix gives the name of the countries as well as their acronyms we will use in the sequel. We also report the $p-values$ of the ADF tests in the model with an intercept only as well as in the model with an intercept and a deterministic time trend. The number of lags in the augmented test is chosen according to a SBC. With the exception of Austria when a deterministic trend is included in the ADF regression we do not reject the null of a unit root at a 5% significance level for none of these countries. It should be noticed from the outset (see Table 4) that a multivariate strategy (VAR, cointegration,...) is quite difficult because series start at a different period. For instance, although the first observation is for 1980Q1 for Belgium or The Netherlands, it is only
1995Q1 for Cyprus or Portugal. Thus if one reduces the variables to a common sample, only few
variables may be jointly considered. Table 4 also reports for each series ARMA models identified
thanks to the SBC in the Hannan-Rissanen approach.

We start by computing in Table 2 the distance in the AR(∞) framework $d_{(i,j)}^{M_1}$. For obvious
practical reasons we have decided to take a value less then ∞ and we choose AR(18). The robustness
of the results to various values of the lag order is an interesting issue, but we leave it for further
research. We obtain $D^{M_1}$ is a $25 \times 25$ matrix with 0 on the diagonal. Table 2 gives for each of
the 25 columns the rows that yields the smallest value in that column. In some cases there exists
a symmetry. For instance Italy gives the smallest value of the distance for the column of Belgium
and vice versa if we go to the column of Italy. Sometimes this is not the case: For UK, Italy gives
the smallest value but UK gives the smallest value for Germany.

Let us first comment the first few steps before giving the whole picture. It emerges from Table
2 that the relationship that gives the smallest distance is the one between Italy and Belgium. We
have a symmetry here in the sense that the smallest value in the column for Belgium is Italy and the
other way around. We average Italy and Belgium, two growth rates observed on the same sample.
Next we compute the distance matrix again between this new construct and the 23 remaining
growth rates. The best relationship is the one between the Italy-Belgium aggregate and Latvia.
However, it is reported in Table 2 that the bivariate relationship between United kingdom and
Germany is better. We consequently consider that pair that we aggregate. In the next round we
have two aggregates and the remaining 21 countries. The best relationship is the one between
Belgium-Italy with Latvia. We average Belgium, Italy and Latvia. Next the minimum distance is
obtained for the relation between the two aggregates that we average, next France is added to that
set, and so on until the last country.

Table 3 gives a complete picture of the successive steps. We also report the maximum ARMA
orders in the respective sets, an information available from Table 4 in the appendix. Column
ARMA$^{SCB}$ in Table 3 reports the ARMA obtained on the aggregate using the SCB in the Hannan-
Rissanen approach. An aggregate is potentially considered if the ARMA orders obtained for the
aggregate is less or equal to the series that compose that aggregate. We see that it is always the
case in Table 3. However and because it could be that SCB is too parsimonious we also report
the $p-value$ of LM test of the null of no autocorrelation (from 1 to 5 quarters) when an ARMA
model is estimated on aggregates. This helps to cut off the set of countries. Indeed, after the step
18, the $p-value$ of the LM test seriously decreases. Also the AR(1) coefficient decreases from 0.5
to less than 0.3 if we include Greece and next countries while it was stable until that. Consequently
Table 2: Pairwise initial minimum distances for each column of the distance matrix

<table>
<thead>
<tr>
<th>columns</th>
<th>rows</th>
<th>$d_{(i,j)}^{M_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Belgium</td>
<td>Italy</td>
<td>0.4089</td>
</tr>
<tr>
<td>Italy</td>
<td>Belgium</td>
<td>0.4089</td>
</tr>
<tr>
<td>United Kingdom</td>
<td>Italy</td>
<td>0.4488</td>
</tr>
<tr>
<td>Germany</td>
<td>United Kingdom</td>
<td>0.4717</td>
</tr>
<tr>
<td>Latvia</td>
<td>Italy</td>
<td>0.4940</td>
</tr>
<tr>
<td>Finland</td>
<td>United Kingdom</td>
<td>0.5357</td>
</tr>
<tr>
<td>France</td>
<td>United Kingdom</td>
<td>0.5860</td>
</tr>
<tr>
<td>Poland</td>
<td>Italy</td>
<td>0.6080</td>
</tr>
<tr>
<td>The Netherlands</td>
<td>United Kingdom</td>
<td>0.6137</td>
</tr>
<tr>
<td>Cyprus</td>
<td>Belgium</td>
<td>0.6374</td>
</tr>
<tr>
<td>Denmark</td>
<td>Germany</td>
<td>0.6383</td>
</tr>
<tr>
<td>Spain</td>
<td>Finland</td>
<td>0.6955</td>
</tr>
<tr>
<td>Lithuania</td>
<td>Finland</td>
<td>0.7747</td>
</tr>
<tr>
<td>Slovenia</td>
<td>The Netherlands</td>
<td>0.7937</td>
</tr>
<tr>
<td>Portugal</td>
<td>Slovakia</td>
<td>0.7940</td>
</tr>
<tr>
<td>Slovakia</td>
<td>Portugal</td>
<td>0.7940</td>
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<td>Denmark</td>
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<td>Lithuania</td>
<td>0.8627</td>
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<td>Hungary</td>
<td>Belgium</td>
<td>0.8965</td>
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<tr>
<td>Sweden</td>
<td>Italy</td>
<td>0.9385</td>
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<tr>
<td>Greece</td>
<td>The Netherlands</td>
<td>1.0119</td>
</tr>
<tr>
<td>Ireland</td>
<td>Greece</td>
<td>1.4709</td>
</tr>
<tr>
<td>Czech Republic</td>
<td>Hungary</td>
<td>1.7447</td>
</tr>
<tr>
<td>Austria</td>
<td>Sweden</td>
<td>10.8323</td>
</tr>
<tr>
<td>Malta</td>
<td>Ireland</td>
<td>11.1944</td>
</tr>
</tbody>
</table>
Figure 1: Growth rate of an EU aggregate: 18 countries

Figure 2:

we consider an aggregate with \{be, it, uk, de, lv, fr, pl, sp, fi, nl, cy, sk, dk, ee, lt, hu, se\}, a result that was difficult to obtain otherwise. Figure 1 gives the aggregate which is estimated by

\[ \Delta \overline{EU}_{18,t} = 0.00344 + 0.5157 \Delta EU_{18,t-1} \]

with \( R^2 = 0.29 \). We do not reject the null of linearity, homoskedasticity, stability (Andrews Chow test), ARCH. For instance the maximum of the sequential Chow test is for the period 1994Q1, when data for additional countries are present, but the \( p-value \) of that test is 0.47. Normality is the only problem but this is due to negative extreme values.

It should be easy now to apply a dating procedure such as the BBQ approach. For instance and considering a window of 5 quarters, Figure 2 gives the contraction period. It is noticed that there are only a few classical cycles for the Euro indicators as compared to the US for instance. This may justify why the growth cycle definition is preferred by European authorities. Figures 5 and 6 illustrate the use of the HP and the BK filter.
Table 3: Minimum distances by columns of $d$

<table>
<thead>
<tr>
<th>Step</th>
<th>Countries</th>
<th>$d_{(i,j)}$</th>
<th>$\max(p_i,q_i)$</th>
<th>ARMA$_{SB}$</th>
<th>$\rho_i$</th>
<th>LM$_{1-5}$</th>
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<tr>
<td>1</td>
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<td>0.408</td>
<td>(1,0)</td>
<td></td>
<td>0.41</td>
<td>0.16</td>
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<tr>
<td>2</td>
<td>uk, de</td>
<td>0.471</td>
<td>(2,0)</td>
<td>(3,0)</td>
<td>0.25,-0.02,0.26</td>
<td>0.63</td>
</tr>
<tr>
<td>3</td>
<td>be, it, lv</td>
<td>0.622</td>
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<td>(1,0)</td>
<td>0.48</td>
<td>0.03</td>
</tr>
<tr>
<td>4</td>
<td>2+3</td>
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<td>(2,1)</td>
<td>(1,0)</td>
<td>0.56</td>
<td>0.26</td>
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<tr>
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<td>(2,1)</td>
<td>(1,0)</td>
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</tr>
<tr>
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<td>(1,0)</td>
<td>0.54</td>
<td>0.19</td>
</tr>
<tr>
<td>7</td>
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<td>0.695</td>
<td>(3,2)</td>
<td>(3,0)</td>
<td>0.05,0.27,0.38</td>
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</tr>
<tr>
<td>8</td>
<td>sp, fi, nl</td>
<td>0.693</td>
<td>(3,2)</td>
<td>(3,0)</td>
<td>0.02,0.25,0.35</td>
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</tr>
<tr>
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<td>8+cy</td>
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<td>(3,2)</td>
<td>(3,0)</td>
<td>0.04,0.23,0.33</td>
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</tr>
<tr>
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<td>(0,0)</td>
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<td>(0,0)</td>
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<td>-</td>
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<td>24</td>
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<td>&gt;10</td>
<td>(4,3)</td>
<td>(1,0)</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>
Figure 3: Level of the EU 18 indicator and classical cycles
Figure 4: Example: Hodrick-Prescott Filter (lambda=1600) and growth cycles

Figure showing EU_18, Trend, and Cycle over time from 1985 to 2005.
Figure 5: Exemple2: Baxter-King filter and growth cycles
6 Conclusion

This paper determines thanks to univariate models, similar series that can be aggregated to form an indicator for business cycle analysis. Because they may be seen as the implied final equations subject to co-movements, nothing is lost compared to the multivariate model. Many questions are still open at this stage of the analysis. First, it must be raised the choice of a metric to compare variables. The one we have illustrated in this paper, the infinite AR, is not a necessary condition for our FE representation but it is a basis of comparison with other measures that are currently under investigation.

Many elements need to be evaluated at the light of a Monte Carlo analysis. For instance what is the most appropriate information criteria for identifying the optimal ARMA models? Is the analysis affected by a different number of observations? What is the best way to average series? Is the simple average roughly correct or must other measures such as the PCA be considered? And in this latter case how do we consider different period of observations? How do outliers influence the final results ARMA models.

7 References


<table>
<thead>
<tr>
<th>Country</th>
<th>$\tau_e$ (k)</th>
<th>$\tau_{ct}$ (k)</th>
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<th>Span</th>
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<td>Be Belgium</td>
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<td>ARMA(0,0)</td>
<td>1980Q1-2007Q2</td>
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<td>Cz Czech Republic</td>
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<td>1995Q1-2007Q2</td>
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<td>Dk Denmark</td>
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<td>De Germany</td>
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<td>Ee Estonia</td>
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<td>ARMA(0,0)</td>
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<td>Ie Ireland</td>
<td>0.23 (1)</td>
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<td>Gr Greece</td>
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<td>ARMA(3,0)</td>
<td>2001Q1-2007Q2</td>
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<td>Nl The Netherlands</td>
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<td>ARMA(0,0)</td>
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</tr>
<tr>
<td>At Austria</td>
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