A Shapley Value for Games with Restricted Coalitions

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Abstract: A "restriction" is a monotonic projection assigning to each coalition of a finite player set \( N \) a subcoalition. On the class of transferable utility games with player set \( N \), a Shapley value is associated with each restriction by replacing, in the familiar probabilistic formula, each coalition by the subcoalition assigned to it. Alternatively, such a Shapley value can be characterized by restricted dividends. This method generalizes several other approaches known in literature. The main result is an axiomatic characterization with the property that the restriction is determined endogenously by the axioms.

1 Introduction

A transferable utility game (or, briefly, game) is a pair \((N, v)\) where \( N = \{1, 2, \ldots, n\} \) (\( n \geq 1 \)) is the set of players and \( v: 2^N \rightarrow \mathbb{R} \) is a map assigning to each coalition \( S \subseteq N \) its worth \( v(S) \), with \( v(\emptyset) = 0 \). Assuming that the grand coalition \( N \) will be formed, the question arises how to divide the worth \( v(N) \) among the players. A well-known answer to this question was provided by Shapley (1953), now known as the Shapley value (to be introduced below). Later authors (e.g., Myerson, 1977; Owen, 1986) realized that in many practical situations not every coalition is formable, including the grand coalition. This may be due to a lack of communication possibilities, or certain institutional constraints. For instance, Gilles et al. (1992) give an example of a game \( \{(1, 2, 3), v\} \) with \( v(1) = 10, v(2) = v(3) = 0, v(12) = 20, v(13) = 30, v(23) = 0, \) and \( v(N) = 30 \). Here player 1 is the seller of an object who values this object at 10 dollars; 2 and 3 are buyers who value the object at 20 and 30 dollars, respectively. Now suppose that player 2 can veto the sale of the object. This gives rise to a new game \( \{(1, 2, 3), w\} \) with \( w(13) = 10 \) and \( w(S) = v(S) \) otherwise. The Shapley value assigns the distribution \( (21\frac{1}{2}, 1\frac{1}{2}, 6\frac{1}{2}) \) to the game \( \{(1, 2, 3), v\} \) and \( (18\frac{1}{2}, 8\frac{1}{2}, 3\frac{1}{2}) \) to the game \( \{(1, 2, 3), w\} \); these distributions reflect the additional power of player 2.

In the present paper these kinds of constraints are modelled by the concept of a restriction, which is defined to be a map \( \rho: 2^N \rightarrow 2^N \) with the following three properties:

1 With no confusion likely, we write \( v(1) \) instead of \( v(\{1\}) \), etc.
2 This part is distinct from the example given by Gilles et al. (1992). For the present purpose the difference is not essential.
(R1) For all $S \in 2^N$, $\rho(S) \subseteq S$.
(R2) For all $S, T \in 2^N$, if $S \subseteq T$ then $\rho(S) \subseteq \rho(T)$.
(R3) For all $S \in 2^N$, $\rho(\rho(S)) = \rho(S)$.

Thus, a restriction is a monotonic projection (R2 and R3, respectively) assigning to each coalition a subcoalition (R1). With each restriction $\rho$ we associate a restricted Shapley value $\varphi^\rho$ by

$$
\varphi^\rho_i(v) = \sum_{S \subseteq S \neq i} \frac{|S|! (n - |S| - 1)!}{n!} [v(\rho(S \cup \{i\})) - v(\rho(S))]
$$

for every $i \in N$. If $\rho$ is the identity, then $\varphi^\rho$ is the familiar Shapley value (Shapley, 1953; Roth, 1988), and we write $\varphi$ instead of $\varphi^\rho$. Obviously, $\varphi^\rho(v) = \varphi(\rho \circ v)$ for every restriction $\rho$ and every game $v$. In the example above, $w = v \circ \rho$ where $\rho(\{1, 3\}) = \{1\}$, $\rho(\{3\}) = \emptyset$, and $\rho(S) = S$ otherwise. Apparently, $\varphi(w) = \varphi(\rho \circ v) = \varphi^\rho(v)$, so the Shapley value of $w$ is obtained by applying an appropriately restricted Shapley value to $v$.

In the next section we provide necessary and sufficient conditions for a class of coalitions to correspond to the image of a restriction. In section 3, the dividend method (Harsanyi, 1959) will be adapted in order to obtain restricted Shapley values. Section 4 presents our main result, which is an axiomatic characterization of restricted Shapley values. Section 5 shows how our approach is related to (and generalizes) various other results in literature.

2 Restrictions

Let $\rho$ be a restriction and let

$$
\text{Im}(\rho) := \{S \in 2^N : S = \rho(T) \text{ for some } T \in 2^N\}
$$

be the image of $\rho$. By R1, $\rho(\emptyset) = \emptyset$. Further, for $S, T \in \text{Im}(\rho)$,

$$
S \cup T = \rho(S) \cup \rho(T) \subseteq \rho(S \cup T) \subseteq S \cup T
$$

where the equality follows from R3, the first inclusion from R2, and the last inclusion from R1. Consequently, $S \cup T = \rho(S \cup T)$, i.e., $S \cup T \in \text{Im}(\rho)$. So the image of a restriction is union-closed and contains the empty set.

Conversely, let $\Omega$ be a subset of $2^N$ with these two properties. Let the map $\rho^\Omega : 2^N \to 2^N$ assign to each $S \in 2^N$ its largest subset in $\Omega$, that is,

$$
\rho^\Omega(S) = \bigcup (\Omega \cap 2^S).
$$

3 With no confusion likely, we write $v$ instead of $(N, v)$. 

It is straightforward to check that \( \rho^\Omega \) is a restriction, and \( \text{Im}(\rho^\Omega) = \Omega \).

Further, for a restriction \( \rho \) and for each \( S \subseteq N \) the set \( \rho(S) \) is contained in \( \text{Im}(\rho) \); and for each subset \( T \) of \( S \) with \( T \subseteq \text{Im}(\rho) \) we have \( T = \rho(T) \cap \rho(S) \) by R2, implying \( \rho(S) = \cup \{ \text{Im}(\rho) \cap 2^S \} = \rho^{\text{Im}(\rho)}(S) \). Therefore, \( \rho = \rho^{\text{Im}(\rho)} \), i.e., \( \rho \) is completely determined by its image.

The following lemma collects these considerations.

**Lemma 2.1**

(i) For any restriction \( \rho \), \( \text{Im}(\rho) \) is union-closed and contains the empty set.

(ii) If \( \Omega \subseteq 2^N \) is union-closed and contains the empty set, then \( \rho^\Omega \) is a restriction.

(iii) For any restriction \( \rho \), \( \rho = \rho^{\text{Im}(\rho)} \).

### 3 Dividends

Let \( \nu \) be a game and \( \Omega \subseteq 2^N \) with \( \emptyset \in \Omega \). Define the map \( \Delta^\Omega_\nu : \Omega \to \mathbb{R} \) recursively by

\[
\Delta^\Omega_\nu(S) = \begin{cases} 
0 & \text{if } S = \emptyset \\
\nu(S) - \sum_{T \subseteq S, T \in \Omega \setminus \{ \emptyset \}} \Delta^\Omega_\nu(T) & \text{if } S \in \Omega \setminus \{ \emptyset \}.
\end{cases}
\]

The number \( \Delta^\Omega_\nu(S) \) is interpreted as the additional dividend the coalition \( S \) obtains if it is formed, given that all proper formable \((e \in \Omega)\) subcoalitions have been formed already. For \( \Omega = 2^N \), \( \Delta_\nu \) coincides with the dividends introduced by Harsanyi (1959); we write \( \Delta_\nu \) instead of \( \Delta^\Omega_\nu \). For each \( T \subseteq N \), let \( u_T \) denote the unanimity game on \( T \), i.e., \( u_T(S) = 1 \) if \( T \subseteq S \), \( u_T(S) = 0 \) otherwise. It is well-known that the space of all games with player set \( N \) is a \( 2^N - 1 \) dimensional vector space with \( \{ u_T : T \neq \emptyset \} \) as a basis. Furthermore, we have (Harsanyi, 1959):

\[
\nu = \sum_{T \subseteq N} \Delta_\nu(T) u_T \tag{3.1}
\]

for every game \( \nu \). It turns out that for \( \rho \)-restricted games we have a similar expression in terms of restricted dividends. This is implied by the following lemma.

**Lemma 3.1:** Let \( \nu \) be a game and \( \rho \) a restriction, and let \( \Omega = \text{Im}(\rho) \). Then

\[
\Delta_{\nu^{\rho}}(S) = \begin{cases} 
0 & \text{if } S \notin \Omega \\
\Delta^\Omega_\nu(S) & \text{if } S \in \Omega
\end{cases}
\]

for each \( S \in 2^N \).
\textbf{Proof:} The proof is by induction on the number of players in $S$. If $S = \emptyset$, then $S \in \Omega$ and $\Delta_{\nu \circ \rho}(\emptyset) = 0 = \Delta_\rho^0(\emptyset)$. Suppose the lemma holds for all coalitions with at most $k$ players ($0 \leq k < n$), and let $S$ be a coalition with $k + 1$ players. Then

\begin{equation}
\Delta_{\nu \circ \rho}(S) = \nu \circ \rho(S) - \sum_{T \subseteq S, \, T \neq S} \Delta_{\nu \circ \rho}(T). \tag{3.2}
\end{equation}

Suppose first that $S \notin \Omega$, i.e., $\rho(S) \neq S$. Observe that $T \in 2^S \setminus \Omega$ implies $T \subseteq \rho(S)$ by R2 and R3, and that $T \subseteq \rho(S)$ implies $T \subseteq S$ by R1. Together with the induction hypothesis we derive

\begin{align*}
\sum_{T \subseteq S, \, T \neq S} \Delta_{\nu \circ \rho}(T) &= \sum_{T \subseteq S, \, T \in \Omega} \Delta_{\nu \circ \rho}(T) \\
&= \sum_{T \subseteq \rho(S), \, T \in \Omega} \Delta_{\nu \circ \rho}(T) \\
&= \sum_{T \subseteq \rho(S), \, T \in \Omega} \Delta_\rho^0(T) \\
&= \nu(\rho(S)).
\end{align*}

Together with (3.2) this proves that $\Delta_{\nu \circ \rho}(S) = 0$.

Next, suppose $S \in \Omega$. Then

\begin{align*}
\Delta_{\nu \circ \rho}(T) &= \nu(\rho(S)) - \sum_{T \subseteq S, \, T \notin \Omega \setminus \{S\}} \Delta_{\nu \circ \rho}(T) \\
&= \nu(S) - \sum_{T \subseteq S, \, T \notin \Omega \setminus \{S\}} \Delta_\rho^0(T) \\
&= \Delta_\rho^0(S).
\end{align*}

\[ \square \]

Lemma 3.1 and formula (3.1) immediately imply the following corollary, which will be used in the next section.

\textbf{Corollary 3.2: For every game $\nu$ and restriction $\rho$,}

\[ \nu \circ \rho = \sum_{T \in \text{Im}(\rho)} \Delta_{\nu \circ \rho}(T) |T|^{-1}. \]

Another well-known fact is the following formula for the Shapley value $\varphi$. Let $\nu$ be a game, then for every $i \in N$,

\[ \varphi_i(\nu) = \sum_{S \subseteq N, \, i \in S} |S|^{-1} \Delta_{\nu}(S), \tag{3.3} \]

i.e., the Shapley value assigns to each player the sum of his proportion of the dividends obtained by those coalitions of which he is a member. This can be derived from (3.1) and the definition (and in particular the linearity) of the Shapley value. Formula (3.3) and Lemma 3.1 obviously imply
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\[ \varphi_i^\rho(u) = \varphi_i(u \circ \rho) = \sum_{S: i \in S, S \subseteq \text{im}(\rho)} |S|^{-1} \Delta_{\text{im}(\rho)}^i(S) \]

for every restriction \( \rho \) and player \( i \in N \). This provides an alternative expression for restricted Shapley values. Note that this expression uses only coalitions in the image of \( \rho \). This is in contrast with the definition of a restricted Shapley value in Section 1, where all coalitions play a role.

4 An Axiomatization

In this section we present an axiomatic characterization of restricted Shapley values. The axioms will be formulated for a so-called value \( \psi: \mathcal{G}^N \rightarrow \mathbb{R}^N \), where \( \mathcal{G}^N \) denotes the class of all transferable utility games with player set \( N \). A restriction \( \rho \) will not be exogenously given, but endogenously determined by the axioms.

Let \( S \) be a coalition. Call \( S \) \( \psi \)-essential if \( S = \emptyset \) or for all \( u, v \in \mathcal{G}^N \) with \( u(T) = v(T) \) for all \( T \neq S \) we have: if \( u(S) \neq v(S) \) then \( \psi(u) \neq \psi(v) \). Call \( S \) \( \psi \)-inessential if \( S \neq \emptyset \) and for all \( u, v \in \mathcal{G}^N \) with \( u(T) = v(T) \) for all \( T \neq S \) we have: \( \psi(u) = \psi(v) \). A player \( i \in N \) is a veto player in a game \( u \) if \( u(S) = 0 \) for every \( S \) with \( S \ni i \).

Observe that the worth of a nonempty \( \psi \)-essential coalition always matters for the distribution \( \psi(u) \), whereas the worth of a \( \psi \)-inessential coalition never matters. The empty set is defined to be \( \psi \)-essential just for convenience; it guarantees the existence of maximal \( \psi \)-essential coalitions, i.e., \( \psi \)-essential coalitions \( S \) such that no proper superset of \( S \) is \( \psi \)-essential.

The axioms are as follows, for a value \( \psi \).

(A1) Every coalition \( S \) is \( \psi \)-essential or \( \psi \)-inessential.
(A2) If \( S \) and \( T \) are \( \psi \)-essential coalitions, then so is \( S \cup T \).
(A3) For every \( u \in \mathcal{G}^N \), \( \sum_{i \in S} \psi_i(u) = u(S) \) for some maximal \( \psi \)-essential coalition \( S \).
(A4) For all \( u, v \in \mathcal{G}^N \) and \( i \in N_i \), if for all \( S, T \) with \( T \psi \)-essential, \( i \in T \) and \( S \) a maximal \( \psi \)-essential subcoalition of \( T \setminus \{i\} \) we have \( u(T) = v(S) \geq w(T) = u(S) \), then \( \psi_i(u) = \psi_i(v) \).
(A5) For every \( u \in \mathcal{G}^N \) and \( i, j \in N_i \), if \( i \) and \( j \) are veto players in \( u \), then \( \psi_i(u) = \psi_j(u) \).

Axiom A1 excludes the existence of a coalition whose worth is taken into account by \( \psi \) in one game but not in some other game. Axiom A2 states that if the worths of coalitions \( S \) and \( T \) always matter for the determination of \( \psi \), then so should the worth of the union \( S \cup T \). Axiom A3 is a generalization of the usual efficiency axiom. Similarly, axiom A4 generalizes the monotonicity axiom proposed by Young (1985). Axiom A5, finally, states that veto players in a game should be treated as equally powerful.
We will demonstrate that these five axioms characterize the class of restricted Shapley values. The proof of this result is inspired by the characterization of the Shapley value by Young (1985). We introduce the notation

\[ \gamma_S = \frac{|S|! (n - |S| - 1)!}{n!} (S \subseteq N) \]

for the coefficients in the definition of the restricted Shapley values \( \varphi^\rho \).

**Proposition 4.1:** Let \( \rho \) be a restriction. Then \( \varphi^\rho \) satisfies A1–A5.

**Proof**

(A1) It is obvious from the definition of \( \varphi^\rho \) that every coalition \( T \) with \( \rho(T) \neq T \) is \( \varphi^\rho \)-inessential. Next, let \( i \in S \in \text{Im}(\rho) \). Let \( \nu \in \mathcal{B}^N \). Then

\[ \varphi_i^\rho (\nu) = \sum_{T : i \in T} \gamma_T [\nu(\rho(T \cup \{i\})) - \nu(\rho(T))]. \quad (4.1) \]

Because \( i \notin T \) and \( \rho(T) \subseteq T \) by R1, \( \rho(T) \neq S \). Hence, \( \nu(\rho(T)) \) never changes if we vary \( \nu(S) \). Further, \( \nu(\rho(T \cup \{i\})) \) changes if and only if \( \rho(T \cup \{i\}) = S \) and in particular for \( T = S \setminus \{i\} \). Therefore, \( \varphi_i^\rho (\nu) \) changes whenever \( \nu(S) \) as varied. So \( S \) is \( \varphi^\rho \)-essential.

(A2) \( S, T \) are \( \varphi^\rho \)-essential if and only if \( S, T \in \text{Im}(\rho) \) (cf. the proof of A1), and then \( S \cup T \in \text{Im}(\rho) \), by Lemma 2.1. It follows (by the proof of A1) that \( S \cup T \) is \( \varphi^\rho \)-essential.

(A3) This follows by noting that \( \sum_{i \in \mathcal{I}} \varphi_i^\rho (\nu) = \sum_{i \in \mathcal{I}} \varphi_i(\nu \circ \rho) = \nu(\rho(N)) \) for every \( \nu \in \mathcal{B}^N \) by the efficiency of the Shapley value, and that \( \rho(N) \) is the maximal \( \varphi^\rho \)-essential coalition (see the proof of A1).

(A4) Consider again the expression for \( \varphi_i^\rho (\nu) \) in (4.1). If \( i \notin \rho(T \cup \{i\}) \), then \( \rho(T \cup \{i\}) = \rho(T) \), and the expression between the brackets vanishes, independent of the game \( \nu \). Otherwise, \( i \in \rho(T \cup \{i\}) \) and \( i \notin \rho(T \subseteq \rho(T \cup \{i\}) \), so for \( \nu \) and \( w \) as in the formulation of A4, we have \( \varphi_i^\rho (\nu) \geq \varphi_i^\rho (w) \), as desired.

(A5) Suppose \( i \) and \( j \) are veto players in the game \( \nu \). Then

\[ \varphi_i^\rho (\nu) = \sum_{T : i \notin T} \gamma_T [\nu(\rho(T \cup \{i\}))] = \sum_{T : i \notin T} \gamma_T [\nu(\rho(T \cup \{i\}))] = \sum_{T : i \notin T} \gamma_T [\nu(\rho(T \cup \{i, j\}))] \]

and similarly \( \varphi_j^\rho (\nu) = \sum_{T : j \notin T} \gamma_T [\nu(\rho(T \cup \{i, j\}))] \).

\( \square \)

**Proposition 4.2:** Let \( \psi \) be a value satisfying A1–A5. Then there exists a unique restriction \( \rho \) such that \( \psi = \varphi^\rho \).
Proof: Let $\Omega = \{ S \in 2^N : S \text{ is } \psi\text{-essential}\}$. Then $\emptyset \in \Omega$ and by A2, $\Omega$ is union-closed, so by Lemma 2.1, $\varphi^\Omega$ is a restriction. Note that $\rho$ assigns the maximal $\psi$-essential subset to every $S \in 2^N$. We first prove that $\psi = \varphi^\rho$.

(i) Let $z$ denote the zero game, $z(S) = 0$ for every $S \in 2^N$. Note that every player is a veto player in $z$, so by A5: $\psi_i(z) = \psi_j(z)$ for all $i, j \in N$. By A3, $\sum_{i \in N} \psi_i(z) = z(\rho(N)) = 0$. Hence, $\psi(z) = (0, 0, \ldots, 0)$.

(ii) Let $T$ be a $\psi$-essential coalition, and let $u_T$ be the corresponding unanimity game. Let $c \in \mathbb{R}$.

Suppose $i \in T$. Let $S$ and $S'$ be $\psi$-essential with $i \in S$ and $S \subset S \setminus \{i\}$, $S'$ maximal. If $T \subset S$, then also $T \subset S'$ by maximality of $S'$, the fact that $\Omega$ is union-closed, and $i \notin T$. If $T \subset S$ then obviously $T \subset S'$. Thus, we always have $c u_T(S) - c u_T(S') = 0 = z(S) - z(S')$. By A4 and (i), we obtain $\psi_i(c u_T) = \psi_i(z) = 0$.

Suppose $i \notin T$. Then $i$ is a veto player in $c u_T$. By A5, $\psi_i(c u_T) = \psi_i(c u_T)$ for all $j \in T$. By A3, $\sum_{j \in N} \psi_j(c u_T) = c u_T(\rho(N)) = c$ since $T \subset \rho(N)$ by R2.

We conclude that $\psi(c u_T) = \varphi^\rho(c u_T) = c e^T = 1/T_1$ for every $\psi$-essential coalition $T$ ($e^T$ denotes the vector in $\mathbb{R}^N$ with $e^T_i = 1$ if $i \in T$, $e^T_i = 0$ if $i \notin T$).

(iii) Let $v$ be a game, $v = \sum \alpha_T u_T$ with $\alpha_T \neq 0$ on the basis of unanimity games. We show that $\psi(v) = \varphi^\rho(v)$ by induction on the number of unanimity games in the representation $\sum \alpha_T u_T$. Because $\varphi^\rho(v) = \varphi^\rho(v \circ \rho)$ and, by A1, $\psi(v) = \psi(v \circ \rho)$, by Corollary 3.2 it is without loss of generality to assume that this representation contains only unanimity games corresponding to $\psi$-essential coalitions. If the representation contains exactly zero or one unanimity games, then the desired equality follows from (i) or (ii) above. Now assume $\psi(v) = \varphi^\rho(v)$ whenever there are less than $k$ ($k \geq 2$) unanimity games in the representation of $v$, and assume $v = \sum_{r=1}^k \alpha_{T_r} u_{T_r}$. Let $D = \bigcap_{r=1}^k T_r$. For $i \notin D$, define $w^i = \sum_{r \in T_r \setminus T_i} \alpha_{T_r} u_{T_r}$. By the induction hypothesis,

$$\psi(w^i) = \varphi^\rho(w^i).$$

(4.2)

Let $S, S'$ be coalitions with $i \in S$, $S \subset S \setminus \{i\}$ maximal $\psi$-essential, and $S$ $\psi$-essential. If $T \neq S$ for some $r$ then either $T \subset S$ in which case $T \subset S \setminus \{i\}$, hence $T \subset S'$ by maximality of $S'$, or $T \not\subset S$ in which case obviously $T \not\subset S'$. Thus, we have

$$v(S) - v(S) = \sum_{r=1}^k \alpha_{T_r} u_{T_r}(S) - \sum_{r=1}^k \alpha_{T_r} u_{T_r}(S')$$

$$= \sum_{r \in T_r \setminus T_i} \alpha_{T_r} u_{T_r}(S) - \sum_{r \in T_r \setminus T_i} \alpha_{T_r} u_{T_r}(S')$$

$$= w^i(S) - w^i(S'),$$

therefore, $\psi_i(w^i) = \psi_i(v)$ and $\varphi^\rho_i(w^i) = \varphi^\rho_i(v)$ by A4 for $\psi$ and $\varphi^\rho$. So by (4.2):

$$\psi_i(v) = \psi_i(w^i) = \phi^\rho_i(w^i) = \phi^\rho_i(v)$$

for every $i \notin D$. (4.3)
By (4.3) and A3 for \( \varphi \) and \( \varphi^\rho \), we have
\[
\sum_{i \in D} \psi_i(v) = \sum_{i \in D} \varphi^\rho_i(v).
\] (4.4)

Obviously, every \( i \in D \) is a veto player in \( v \). By A5, \( \psi_i(v) = \psi_j(v) \) and \( \varphi_i^\rho(v) = \varphi_j^\rho(v) \) for all \( i, j \in D \). Thus, by (4.4),

\[
\psi_i(v) = \varphi_i^\rho(v) \text{ for every } i \in D.
\]

Together with (4.3) this implies that \( \psi = \varphi^\rho \).

For uniqueness, suppose that \( \psi = \varphi^\rho = \varphi^{\rho'} \) for restrictions \( \rho \) and \( \rho' \). With \( \Omega \) as above, obviously \( \Omega = \text{Im}(\rho) = \text{Im}(\rho') \) since \( \text{Im}(\rho) \) and \( \text{Im}(\rho') \) are the collections of \( \varphi^\rho \)- and \( \varphi^{\rho'} \)-essential coalitions, respectively (see the proof of A1 in Proposition 4.1). Then Lemma 2.1 (iii) implies \( \rho = \rho' \). \( \square \)

By combining Propositions 4.1 and 4.2 we obtain our main result.

**Theorem 4.3:** A value satisfies axioms A1–A5 if and only if it is a restricted Shapley value.

### 5 Related Literature

Our approach is related to several other ones in literature. Gilles et al. (1992) define a *permission structure* as a map \( F: N \rightarrow 2^N \) with the property that if \( j \in F(i) \) then \( i \notin F(j) \) for all \( i, j \in N \). \( F(i) \) is interpreted as the subset of immediate "subordinates" or "followers" of player \( i \). Let \( \Omega_F^i \) denote the collection of those coalitions \( S \) with the property that if \( i \in S \), then also every \( j \) with \( i \notin F(j) \) is in \( S \), for every \( i \in N \). Further, let \( \Omega_F^i \) denote the collection of those coalitions \( S \) with the property that if \( i \in S \), then at least one \( j \in N \) with \( i \notin F(j) \) is in \( S \) (if such a \( j \) exists), for every \( i \in N \) (see Gilles and Owen, 1991). It is easily established that both \( \Omega_F^i \) ("c" for "conjunctive") and \( \Omega_F^i \) ("d" for "disjunctive") are union-closed and contain the empty set, so that according to Lemma 2.1 both can be described as the image of a restriction. Also the variant of the Shapley value proposed by Gilles et al. (1992) coincides with our concept of a restricted Shapley value. Their Shapley value can be obtained axiomatically by strengthening the system A1–A5 in section 4. For instance, by requiring instead of A2 that the collection of \( \psi \)-essential coalitions be a discerning lattice (see Derks and Gilles, 1992) the class of Shapley values corresponding to restrictions \( \rho \) with \( \text{Im}(\rho) = \Omega_F^i \) for acyclic permission structures \( F \) would be obtained.

Faigle and Kern (1992) consider cooperative games under precedence constraints, which is equivalent to the conjunctive approach described above with acyclic permission structure. They define and axiomatize a Shapley value for such games, which does not coincide with our restricted Shapley value. For instance, their Shapley value does not have to be symmetric for the players of an essential coalition.
A Shapley Value for Games with Restricted Coalitions

$T$ (i.e., $T \in \Omega_2$) in the unanimity game $u_7$, but may reflect the asymmetries inherent in the permission structure $F$.

Our method can also be used to define a Shapley value for so-called multi-choice games, as introduced by Hsiao and Raghavan (1990). A multi-choice game is a refinement of a transferable utility game in the sense that each player $i$ has a finite number of "activity levels" $a_i^0 < a_i^1 < \ldots < a_i^m$. Level $a_i^0$ is interpreted as "not present". A coalition is an $n$-tuple of activity levels, one for each player, and each coalition is assigned a real number, its worth. One can regard each activity level of each player as a separate player and define a permission structure $F$ by $F(a_i^j) = \{a_i^{j+1}\}$ for every $i$ and every $0 \leq j < m$, $F(a_m^i) = \emptyset$ for every $i$. This $F$ is acyclic and $\Omega_2$ contains exactly those "coalitions" which for each activity level of a member also contains all lower activity levels. Such a "coalition" for the extended player set is given the worth of the corresponding coalition in the multi-choice game, obtained by taking for each player his highest activity level occurring in the "coalition" within the extended player set. As before, $\Omega_2$ is the image of a restriction on the extended player set, to which a restricted Shapley value can be associated. Hsiao and Raghavan (1990) also propose a Shapley value for multi-choice games. Again, theirs is different from ours. Consider a unanimity game, which in this context is determined by a vector $\delta \in \bar{A} := \times_{i=1}^n \{a_i^0, \ldots, a_i^m\}$ such that a coalition $b \in A$ has worth 1 if $b \geq \delta$, 0 otherwise. A Shapley value as proposed by Hsiao and Raghavan (1990) distributes the worth of the grand coalition (which equals 1) among the levels or "players" occurring in $\delta$, and, in an equal fashion, to all $n$-tuples of higher levels. Our restricted Shapley value distributes 1 among the levels not exceeding those occurring in $\delta$. In Hsiao and Raghavan's model (and interpretation) each player can only be present at one specific level. In our adaptation of their model (which is equivalent to the adaptation proposed by Faigle and Kern, 1992) the presence of a player at a certain level implies the presence of that player at all lower levels. This distinction largely accounts for the distinction between the respective Shapley values. In fact, Hsiao and Raghavan's Shapley value may be transformed into our restricted Shapley value for this special context of multi-choice games.

Finally, the weighted ordered partitions on which Kalai and Samet (1987) and Nowak and Radzik (1991) base their weighted Shapley values cannot be obtained as the image of a restriction: monotonicity (R2) will be violated. Also the class of values recently characterized by Dragan (1992) does not contain our class of restricted Shapley values, since the latter in general do not satisfy the carrier axiom used by Dragan.

References


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4 This requires an appropriate choice of weights for the Hsiao-Raghavan Shapley value.

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