Range Convexity, Continuity, and Strategy-Proofness of Voting Schemes

HANS PETERS AND HANS VAN DER STEL
University of Limburg, Department of Mathematics, P.O. Box 616, 6200 MD Maastricht, The Netherlands

TON STORCKEN
University of Limburg, Department of Economics, P.O. Box 616, 6200 MD Maastricht, The Netherlands

Abstract: A voting scheme assigns to each profile of alternatives reported by n individuals a compromise alternative. A voting scheme is strategy-proof if no individual is better off by lying, i.e., not reporting a best alternative. In this paper the main results concern the case where the set of alternatives is the Euclidean plane and the preferences are Euclidean. It is shown that for strategy-proof voting schemes continuity is equivalent to convexity of the range of the voting scheme. Using a result by Kalm and Roush (1984), this leads to characterizations of surjective or unanimous, anonymous, strategy-proof voting schemes.

Furthermore, the paper contains an extensive discussion of related results in the area.

1 Introduction

A central problem of the theory of social choice is the aggregation of individual preferences on a given set of alternatives into a group choice. The existence of some kind of regulating institution is presumed; individuals are assumed to obey an aggregation rule employed by such an institution. Since, however, the individual preferences are usually private information, the question arises how to elicit the true preferences from the individuals. One possibility is the use of strategy-proof aggregation rules. Briefly, an aggregation rule is strategy-proof if no individual is better off by lying about his preferences. Strategy-proofness of aggregation rules is a desirable condition, not just for the moral objective of preventing manipulative behavior, but also — and perhaps more importantly — because right decisions are more likely to be taken on the basis of the right information.

The first theoretical results about the existence of strategy-proof aggregation rules, the negative results of Gibbard (1973) and Satterthwaite (1975), were not very encouraging. As Satterthwaite (1975) showed, under Pareto optimality and in the classical discrete framework of social choice theory with all linear or weak
orders allowed, the strategy-proofness condition is logically equivalent to Arrow's (1963) independence of irrelevant alternatives condition, and therefore impossibility results were to be expected.

The issue of strategy-proofness was brought to new life by the work of Laffond (1980) and Moulin (1980). These authors consider continuum domains of alternatives — for instance, the real line in Moulin (1980) — and restricted domains of admissible preferences; in particular, they consider preferences that are single-peaked, a notion that goes back to Black (1948). In these works, as well as in many subsequent papers, nonconstant nondictatorial strategy-proof aggregation rules were characterized in various contexts. Furthermore, this approach made the theory of social choice more readily applicable to standard microeconomic or political models.

The aggregation rules characterized by the mentioned authors are voting schemes: In a voting scheme an individual reports to the institution not a preference relation over the alternatives, but one alternative. The voting scheme aggregates the reported alternatives into a compromise alternative.

In Moulin (1980) the set of alternatives is the whole real line. Each individual reports a point; only single-peaked preferences are allowed, i.e., preferences with a unique best element and decreasing when moving away from this element in either of the two directions. Moulin considers three conditions on voting schemes: strategy-proofness, anonymity, and Pareto optimality. He shows that any voting scheme satisfying the first two conditions must be of the following form. Choose \( n + 1 \) points \( c_1, c_2, \ldots, c_{n+1} \) in \( \mathbb{R} \cup \{-\infty, \infty\} \). If the \( n \) individuals report the points \( x_1, x_2, \ldots, x_n \), then the voting scheme assigns the median point of the \( 2n + 1 \) points \( c_{1}, \ldots, c_{n+1}, x_1, \ldots, x_n \) — that is, the middle point after all points have been arranged in nondecreasing order. From this, it follows that if Pareto optimality is added, the class of voting schemes satisfying the three properties is the subclass of voting schemes determined by \( n - 1 \) points instead of \( n + 1 \). The latter result still holds if only Euclidean preferences are considered, as follows from Proposition 4 in Border and Jordan (1983). In a Euclidean preference, one point is preferred to another if it is closer to the best point, measured according to Euclidean distance. So a Euclidean preference is completely characterized by its best point. Kim and Roush (1981) provide an alternative axiomatic characterization of the same class of solutions.

Laffond (1980) gives a detailed analysis of the case of two agents with Euclidean or quadratic preferences on the higher dimensional Euclidean space. He characterizes the class of anonymous, strategy-proof, surjective solutions by means of so-called option sets (Barberà and Peleg, 1990), which describe the solution points attainable by one agent if the preference(s) of the other agent(s) are being held fixed.

Kim and Roush (1984) and Peters et al. (1992a) also extend Moulin's results to higher dimensions. Kim and Roush characterize all continuous, anonymous, and strategy-proof voting schemes with the Euclidean plane as set of alternatives and only Euclidean preferences allowed. Formally, these voting schemes are described as follows. Choose a pair of orthogonal axes in the plane. Coordinates
of points are projections on these axes. Choose \( n + 1 \) fixed points in \( (\mathbb{R} \cup \{-\infty, \infty\})^2 \). For each profile of individually reported points \( x^1, \ldots, x^n \), add these fixed points and determine the two median coordinates; these coordinates are the coordinates of the point assigned by the voting scheme. Thus, to every choice of a pair of orthogonal axes and of \( n + 1 \) fixed points, there corresponds a continuous, anonymous, and strategy-proof voting scheme, and vice versa. By adding Pareto optimality to these three conditions, Kim and Roush characterize the class of voting schemes defined by taking medians coordinatewise; there are no fixed points, implying that the number of individuals must be odd. (Alternatively formulated, there is an even number of fixed points with for each coordinate half of the numbers equal to \( \infty \) and half of the numbers equal to \(-\infty\).) In Peters et al. (1992a) this same class is characterized without the continuity condition. It is also shown that, in all other cases – dimension higher than two, or an even number of individuals – a solution satisfying anonymity, Pareto optimality and strategy-proofness does not exist.

Another closely related paper is Border and Jordan (1983), already mentioned above. In their main result, these authors characterize the class of all voting schemes that are unanimous and strategy-proof with respect to the class of separable quadratic preferences. Therefore, their strategy-proofness condition (which they call "straightforwardness") is stronger than the one imposed by Kim and Roush (1984) or Peters et al. (1992a). In particular, the coordinate system with respect to which their preferences are defined determines the coordinate system with respect to which the characterized voting schemes are defined whereas, in the other mentioned papers, every voting scheme determines its own orthogonal basis by which it can be described. Chichilnisky and Heal (1981) adopt the same framework as Border and Jordan (1983), but replace the condition of unanimity of a solution by the requirement that its range be rectangular. In Peters et al. (1991) the work of Border and Jordan is generalized to other domains of alternatives and other domains of preferences.

Still other work in the same area was done by Bordes et al. (1989), who, in the larger part of their paper, concentrate on the (stronger) coalitional strategy-proofness condition, and (not surprisingly) on impossibility results. Further, we mention Le Breton and Sen (1991), Barberà et al. (1990), Barberà et al. (1991), and Barberà et al. (1992).

Since, in the discrete context, strategy-proofness is closely related to Arrow's independence of irrelevant alternatives (IIA) condition, the question arises how these conditions are related in models with a continuum of alternatives and under the "tops-only" condition. It turns out that in this context IIA – which concerns the orderings of all alternatives – is much stronger than strategy-proofness – which can be formulated as was done above with respect to the best elements ("tops") of the individual preferences only. For details we refer to Bordes and Le Breton (1989) and Peters et al. (1992b). Moulin (1984) considers the combination of Nash's (1950) and Arrow's IIA for one-dimensional voting schemes.
The present paper is most closely related to Kim and Roush (1984) and Peters et al. (1992a). We consider strategy-proof voting schemes for \( n \) individuals, and with the \( m \)-dimensional Euclidean space as set of alternatives. We concentrate on preferences determined by a unique best point and a metric. Our main results, however, apply to the two dimensional case \((m = 2)\) and the Euclidean metric, i.e., Euclidean preferences. The main result of Kim and Roush (1984), characterizing the class of all strategy-proof, anonymous, continuous voting schemes as the class of all voting schemes obtained by taking medians coordinate wise while adding \( n + 1 \) "fixed" points for each coordinate, is taken as a starting point. As in Peters et al. (1992a) we consider the consequences of replacing the continuity condition by a different axiom. In Peters et al. (1992a) continuity is replaced by Pareto optimality. In the present paper we show that for the two dimensional case continuity of strategy-proof voting schemes is equivalent to convexity of the range of such voting schemes (Theorem 5). Hence, in the result of Kim and Roush (cf. Theorem 1 below) continuity may be replaced by range convexity. As an interesting consequence, it follows that if continuity is replaced by surjectivity (cf. Laffond, 1980; this property is often named "voters' sovereignty" in the social choice literature) or unanimity, the class of voting schemes is the subclass determined by \( n - 1 \) instead of \( n + 1 \) fixed points. Like Pareto optimality these properties have a more direct social choice-theoretic interpretation than the continuity condition.

The rest of the paper is organized as follows. Section 2 contains preliminaries. Sections 3 and 4 treat the one and two individual cases, respectively, and lay the groundwork for the main results. These are formulated and derived in Section 5.

2 Preliminaries

The set of \textbf{individuals} is denoted by \( N := \{1, 2, \ldots, n\} \), where \( n \geq 1 \). These individuals are assumed to have preferences on the Euclidean space \( \mathbb{R}^m \), where \( m \geq 1 \). We assume that each preference is of the following form: There is a point \( x \in \mathbb{R}^m \) (the \textbf{bliss point}) and a metric \( \delta \) on \( \mathbb{R}^m \) such that \( y \) is weakly preferred to \( z \) if and only if \( \delta(y, x) \leq \delta(z, x) \), for all \( y, z \in \mathbb{R}^m \). Moreover, we assume that each preference is determined by one and the same metric. A predominant role is played by the Euclidean metric \( \delta_E \), defined as usual by

\[
\delta_E(x, y) = \sqrt{\sum_{i=1}^{m} (x_i - y_i)^2}.
\]

A \textbf{voting scheme} is a map \( \varphi: (\mathbb{R}^m)^N \rightarrow \mathbb{R}^m \). The idea is that each individual reports a point (for instance, the bliss point of his preference) whereupon \( \varphi \) determines a \textbf{compromise point}. A voting scheme is different from a social choice function, where each individual would report a preference. Of course, given the metric \( \delta \), preferences are completely determined by their bliss points. Elements of \( (\mathbb{R}^m)^N \) are called \textbf{profiles}. 
Preferences may be private information. In order to elicit the individual's true bliss points the following condition may be imposed on a voting scheme. Let $\delta$ be a metric. The voting scheme $\varphi$ is strategy-proof with respect to $\delta$ if for all $i \in N$ and all $p, q \in (\mathbb{R}^m)^N$ with $p(j) = q(j)$ for all $j \neq i$, we have $\delta(p(i), \varphi(p)) \leq \delta(p(i), \varphi(q))$. This means that for every individual it is never advantageous to lie about his true bliss point, whatever the other individuals report.

Other conditions on a voting scheme that play a role in this paper are the following. The voting scheme $\varphi$ is anonymous if $\varphi(p) = \varphi(p \circ \sigma)$ for every permutation $\sigma$ of $N$; in words, the names of the individuals do not matter. We call $\varphi$ range convex if $\varphi((\mathbb{R}^m)^N)$ is a convex set; this is a technical condition which will turn out to be closely related to continuity of strategy-proof voting schemes. $\varphi$ is unanimous if $\varphi(p) = x$ whenever $p(i) = x \in \mathbb{R}^m$ for all $i \in N$. $\varphi$ is surjective if for every $x \in \mathbb{R}^m$ there is a $p \in (\mathbb{R}^m)^N$ with $\varphi(p) = x$. Finally, continuity of $\varphi$ with respect to a metric $\delta$ is defined in the usual way.

The following notations will be used frequently. All occurring points are elements of $\mathbb{R}^m$. By "conv" we denote "the convex hull of". Further, $[x, y] := \text{conv}\{x, y\}$, $[x, y] := [x, y] - \{x, y\}$, $[x, y] := [x, y] - \{y\}$, $[x, y] := [x, y] - \{x, y\}$, $[x, y, z] := [x, y] - \{x, y, z\}$, $[x, y, z] := [x, y] - \{x, y, z\}$, $[x, y, z] := [x, y] - \{x, y, z\}$, $[x, y, z] := [x, y] - \{x, y, z\}$. For a metric $\delta$ and a sequence of points $x_1, x_2, \ldots, x_i \to x$ means $\lim_{i \to \infty} x_i = x$ in the metric $\delta$.

The following lemma states that for a strategy-proof voting scheme, replacing a reported point by the compromise point does not change the compromise point. This corresponds to what is usually called positive association in the social choice literature.

**Lemma 1:** Let $\delta$ be a metric on $\mathbb{R}^m$ and let the voting scheme $\varphi: (\mathbb{R}^m)^N \to \mathbb{R}^m$ be strategy-proof with respect to $\delta$. Let $p, \hat{p} \in (\mathbb{R}^m)^N$ with $\hat{p}(i) \in \{\varphi(p), p(i)\}$ for all $i \in N$. Then $\varphi(\hat{p}) = \varphi(p)$.

**Proof:** It is sufficient to prove the lemma for profiles $p, \hat{p}$ for which there is an $i \in N$ with $\hat{p}(i) = \varphi(p)$ and $\hat{p}(j) = p(j)$ for every $j \neq i$. For such profiles, by strategy-proofness $\delta(\hat{p}(i), \varphi(\hat{p})) \leq \delta(\hat{p}(i), \varphi(p)) = 0$. Hence, $\varphi(\hat{p}) = \varphi(p)$.

In our main results coordinatewise median voting schemes will be central. A collection $\{x^1, x^2, \ldots, x^n\} \subset \mathbb{R}^m$ is a coordinate system in $\mathbb{R}^m$ if $x^i \cdot x^j = 0$ for all $i, j \in \{1, 2, \ldots, m\}$ with $i \neq j$, i.e., if all its elements are orthogonal according to the usual inner product. In other words, such a collection is an orthogonal basis for $\mathbb{R}^m$. Let $k \in \mathbb{N}$ so that $k + n$ is odd, and call a voting scheme $\varphi: (\mathbb{R}^m)^N \to \mathbb{R}^m$ a coordinatewise median voting scheme with $k$ constant points if there exists a coordinate system and points $c^1, c^2, \ldots, c^k \in (\mathbb{R} \cup \{-\infty, \infty\})^m$ so that for every
profile $p \in (\mathbb{R}^n)^N$ and every $j = 1, 2, \ldots, m$,

$$\varphi_j(p) := \text{med}(p(1)_j, p(2)_j, \ldots, p(n)_j, c_j^1, \ldots, c_j^j),$$

where "med" denotes the median of the subsequent real numbers, and all coordinates are expressed with respect to the given coordinate system.

For the case $m = 2$, the following result was derived by Kim and Roush (1984, Theorem 37).

**Theorem 1:** A voting scheme $\varphi: (\mathbb{R}^2)^N \to \mathbb{R}^N$ is anonymous, continuous, and strategy-proof with respect to the Euclidean metric if, and only if, $\varphi$ is a coordinate-wise median voting scheme with $n + 1$ constant points.

The conditions of continuity and range convexity of voting schemes turn out to be closely related. One of our results will be that in Theorem 1 continuity can be replaced by range convexity.

### 3 One Individual

In this section we concentrate on the case of one individual ($n = 1$). By $\varphi$ we mean a voting scheme: $\mathbb{R}^n \to \mathbb{R}^m$, unless stated otherwise. The first result can already be found in Kim and Roush (1984, Theorem 13).

**Theorem 2:** $\varphi$ is continuous and strategy-proof with respect to $\delta_2$ if, and only if, there is a closed convex set $C \subseteq \mathbb{R}^n$ such that $\varphi(x)$ is the nearest point of $C$ to $x$, for all $x \in \mathbb{R}^n$.

That such a nearest point of $C$ in Theorem 2 is indeed unique follows from the following characterization of closed convex sets by Motzkin (1935); see also Valentine (1964, Theorem 7.8).\(^1\)

**Theorem 3:** Let $C \subseteq \mathbb{R}^m$ be a closed (with respect to $\delta_2$) set. Then $C$ is nonempty and convex if, and only if, for every $y \in \mathbb{R}^m$ there is a unique $x \in C$ such that $\delta_2(x, y)$ is minimal.

---

\(^1\) Theorem 3 is formulated here for the Euclidean metric but holds for the wider class of all metrics induced by strictly convex smooth norms.
The main new result in this section in Theorem 4, which will also be used in the next section. Its proof is based on Theorem 3 and the following lemma.

Lemma 2: Let $\delta$ be a metric on $\mathbb{R}^m$. Let $\varphi$ satisfy strategy-proofness with respect to $\delta$. Then the range $C$ of $\varphi$ is a closed (with respect to $\delta$) subset of $\mathbb{R}^m$.

Proof: Let $x^i \in C$ ($i \in \mathbb{N}$), $x \in \mathbb{R}^m$ with $\delta(x, x^i) \to 0$. Since $x^i \in C$, we can define $y^i \in \mathbb{R}^m$ such that $x^i = \varphi(y^i)$ for $i \in \mathbb{N}$. By strategy-proofness, $\delta(x, \varphi(x)) \leq \delta(x, \varphi(y^i))$ for all $i \in \mathbb{N}$. Hence, $\delta(x, \varphi(x)) = 0$, i.e., $\varphi(x) = x$. Therefore, $x \in C$. Consequently, $C$ is closed.

Theorem 4: Let $\varphi$ satisfy strategy-proofness with respect to $\delta_\varphi$. Then $\varphi$ is continuous if and only if $\varphi$ is range convex.

Proof: Let $C \subseteq \mathbb{R}^m$ denote the range of $\varphi$. Lemma 2 implies that $C$ is closed.

First, let $\varphi$ be continuous. In order to prove that $\varphi$ is range convex, it is by Theorem 3 sufficient to prove that for every $y \in \mathbb{R}^m$ there is a unique point of $C$ with minimal distance to $y$. So let $y \in \mathbb{R}^m$ and $u \in C$ with $\delta_\varphi(y, u) = \delta_\varphi(y, x)$ for all $x \in C$. Let $w = \varphi(y)$. Then, by strategy-proofness, $\delta_\varphi(y, w) = \delta_\varphi(y, u)$. If $y = u$ we are done. Otherwise, let $x^i \in [u, y]$ for $i \in \mathbb{N}$ with $x^i \to y$. By strategy-proofness, $\delta_\varphi(x^i, \varphi(x^i)) \leq \delta_\varphi(x^i, u)$ and $\delta_\varphi(y, u) = \delta_\varphi(y, w) \leq \delta_\varphi(y, \varphi(x^i))$ for all $i \in \mathbb{N}$. This is only possible if $\varphi(x^i) = u$ for all $i \in \mathbb{N}$. Hence, $u = \varphi(y)$ by the continuity of $\varphi$. This shows that $u$ must be unique.

Second, let $\varphi$ be range convex. Let $x, x^i \in \mathbb{R}^m$ ($i \in \mathbb{N}$) with $x^i \to x$ (in the metric $\delta_\varphi$) and $w = \varphi(x)$. By strategy-proofness,

$$\delta_\varphi(x, w) \leq \delta_\varphi(x, c) \quad \text{for all } c \in C.$$  \hspace{1cm} (1)

Further,

$$\delta_\varphi(x, \varphi(x^i)) \leq \delta_\varphi(x, x^i) + \delta_\varphi(x^i, \varphi(x^i))$$

$$\leq \delta_\varphi(x, x^i) + \delta_\varphi(x^i, \varphi(x))$$

$$\leq 2\delta_\varphi(x, x^i) + \delta_\varphi(x, w).$$

In view of this we may assume that $\varphi(x^i)$ converges, say to $v$. In particular, we have

$$\delta_\varphi(x, v) \leq \delta_\varphi(x, w).$$  \hspace{1cm} (2)
By strategy-proofness, \( \delta_\varphi(v, \varphi(v)) \leq \lim_{i \to \infty} \delta_\varphi(v, \varphi(x^i)) = 0 \). Hence, \( \varphi(v) = v \). So \( v \in C \). By (1) and (2), \( v \) and \( w \) both minimize the (Euclidean) distance to \( x \) over \( C \). Therefore, by Theorem 3, \( v = w \). Since \( x^i \), \( x \) were arbitrary, this proves continuity of \( \varphi \).

\[ \square \]

Theorem 2 is now easily proved with the help of Theorem 4.

**Proof of Theorem 2:**

i) Let \( \varphi \) satisfy continuity and strategy-proofness with respect to \( \delta_\varphi \). Let \( C = \varphi(\mathbb{R}^m) \). Then by Lemma 2, \( C \) is closed. Theorem 4 implies that \( \varphi \) is range convex. Hence, \( C \) is convex. So by strategy-proofness, \( \varphi(x) \) is a nearest point of \( C \) to \( x \), for all \( x \in \mathbb{R}^m \). By Theorem 3, this point is unique.

ii) Let \( C \subseteq \mathbb{R}^m \) be closed and convex. Let \( \varphi \) be such that \( \varphi(x) \) is a nearest point of \( C \) to \( x \) for all \( x \in \mathbb{R}^m \). Then \( C \) is the range of \( \varphi \). Hence, \( \varphi \) is range convex. So Theorem 4 implies that \( \varphi \) is continuous. By definition \( \varphi \) is strategy-proof with respect to \( \delta_\varphi \).

\[ \square \]

Theorem 2 cannot be generalized for arbitrary metrics, as is shown by the following example.

**Example 1:** Let the metric \( \delta \) on \( \mathbb{R}^2 \) be defined by \( \delta(x, y) := |x_1 - y_1| + |x_2 - y_2| \). Let \( C = \{ x \in \mathbb{R}^2 : x_2 = \frac{1}{2}|x_1| \} \). Let \( \varphi : \mathbb{R}^2 \to \mathbb{R}^2 \) be defined by \( \varphi(x) := (x_1, \frac{1}{2}|x_1|) \) for all \( x \in \mathbb{R}^2 \). Clearly, \( C \) is the range of \( \varphi \). It is not hard to show that, for all \( x \), \( \varphi(x) \) coincides with the unique point in \( C \) for which the distance to \( x \) is minimized. So \( \varphi \) is strategy-proof with respect to \( \delta \). By definition, \( \varphi \) is also continuous. On the other hand, \( C \) is not a convex set.

### 4 Two Individuals

In this section we consider voting schemes \( \varphi : (\mathbb{R}^m)^2 \to \mathbb{R}^m \) which satisfy range convexity and strategy-proofness with respect to the Euclidean metric \( \delta_\varphi \). The results of this section mainly serve as a preparation for the next section, where we focus on the case \( m = 2 \) for an arbitrary number of individuals.

The first lemma establishes conditions under which, for the point reported by the second individual fixed, the distance between the point reported by the first individual and the compromise point assigned by the voting scheme, changes continuously.
Lemma 3: Let $\delta$ be a metric on $\mathbb{R}^m$. Let $\varphi: (\mathbb{R}^m)^2 \to \mathbb{R}^m$ satisfy strategy-proofness with respect to $\delta$. Let $y \in \mathbb{R}^m$. Let $f: \mathbb{R}^m \to \mathbb{R}$ be defined by $f(x) = \delta(x, \varphi(x, y))$ for $x \in \mathbb{R}^m$. Then $f$ is continuous on $\mathbb{R}^m$.

**Proof:** By strategy-proofness,

$$
\delta(x, \varphi(z, y)) - \delta(x, z) \leq \delta(x, \varphi(x, y)) - \delta(x, z)
$$

$$
\leq \delta(x, \varphi(x, y))
$$

$$
\leq \delta(x, \varphi(z, y))
$$

$$
\leq \delta(z, \varphi(z, y)) + \delta(x, z)
$$

for all $x, z \in \mathbb{R}^m$. Hence,

$$
f(z) - \delta(x, z) \leq f(x) \leq f(z) + \delta(x, z)
$$

for all $x, z \in \mathbb{R}^m$. So $\delta(f(z), f(x)) \leq \delta(x, z)$ for all $x, z \in \mathbb{R}^m$. Therefore, $f$ is Lipschitz continuous on $\mathbb{R}^m$. Consequently, $f$ is continuous on $\mathbb{R}^m$. 

Throughout the remainder of this section we make the following assumption.

**Assumption 1:** $\varphi: (\mathbb{R}^m)^2 \to \mathbb{R}^m$ is a voting scheme satisfying range convexity and strategy-proofness with respect to $\delta_E$.

Lemma 4: Let $v, w \in \mathbb{R}^m$ with $\varphi(v, w) = v$ and $\varphi(w, w) = w$. Then $\varphi(x, w) = x$ for all $x \in [v, w]$.

**Proof:** Let $x \in ]v, w[$ and suppose $\varphi(x, w) \neq x$. We will derive a contradiction.

First, define $f: [v, x] \to \mathbb{R}$ by $f(y) := \delta_E(y, \varphi(y, w))$ for all $y \in [v, x]$. By Lemma 3 $f$ is continuous. Let $\bar{x} \in [v, x]$ with $f$ maximal at $\bar{x}$. Since $f(x) > 0$, $f(\bar{x}) > 0$. Also by continuity of $f$, we can take $\bar{x} \in ]v, x[$ with $f(\bar{x}) < f(\bar{x})$ and $f(\bar{x}) > 0$ for all $y \in [\bar{x}, \bar{x}]$. Let $\bar{x} \in [\bar{x}, \bar{x}]$ with $f(\bar{x})$ minimal on $[\bar{x}, \bar{x}]$. So $\bar{x} \neq x$. Suppose $\angle(v, \bar{x}, \varphi(\bar{x}, w)) > 90^\circ$. Then there is an $x^* \in [\bar{x}, \bar{x}]$ with $\delta_E(x^*, \varphi(x, w)) < \delta_E(\bar{x}, \varphi(\bar{x}, w))$. By strategy-proofness, $f(x^*) = \delta_E(x^*, \varphi(x^*, w)) \leq \delta_E(x^*, \varphi(\bar{x}, w)) < \delta_E(\bar{x}, \varphi(\bar{x}, w)) = f(\bar{x})$, contradicting the minimality of $f(\bar{x})$. Hence,

$$
\angle(v, \bar{x}, \varphi(\bar{x}, w)) \leq 90^\circ
$$
This implies that \( \delta_e(w, x) < \delta_e(w, \varphi(x, w)) \). By range convexity and Lemma 1, \( \varphi(x, x) = x \). So

\[
\delta_e(w, \varphi(x, x)) < \delta_e(w, \varphi(x, w))
\]

contradicting strategy-proofness.

---

**Lemma 5**: Let \( x, y \in \mathbb{R}^m \) with \( \varphi(y, y) = y \) and \( \varphi(x, y) \notin \{x, y\} \). Then \( \angle(x, \varphi(x, y), y) \geq 90^\circ \).

**Proof**: Let \( u = \varphi(x, y) \). Suppose \( \angle(x, u, y) < 90^\circ \). Then we can take \( \hat{x} \in [y, u[ \) with \( \delta_e(x, \hat{x}) < \delta_e(x, u) \). Hence by strategy-proofness,

\[
\varphi(x, y) \neq \hat{x}.
\]

(3)

On the other hand, by Lemma 1, \( \varphi(u, y) = u \). Hence, by Lemma 4, \( \varphi(x, y) = x \). This contradicts (3). Consequently, \( \angle(x, \varphi(x, y), y) \geq 90^\circ \).

---

**Lemma 6**: Let \( x, y \in \mathbb{R}^m \) with \( \varphi(x, y) = y \). Then \( \varphi(y, y) = y \) for all \( z \in [y, x, \infty[ \).

**Proof**: Assume \( x \neq y \), otherwise the lemma is trivial. Let \( z \in [y, x, \infty[ \). Let \( w = \varphi(z, y) \). Suppose \( w \neq y \). Then by Lemma 1, \( z \neq y \). If \( w = z \), trivially \( \angle(z, y, w) < 90^\circ \). Otherwise, Lemma 5 implies \( \angle(z, y, w) < 90^\circ \). Therefore, \( \angle(x, y, w) < 90^\circ \). Hence, we can take \( \hat{x} \in [y, w[ \) such that

\[
\delta_e(x, \hat{x}) < \delta_e(x, y).
\]

(4)

By Lemma 1, \( \varphi(w, y) = w \) and \( \varphi(y, y) = y \). Hence, by Lemma 4, \( \varphi(x, y) = x \). Because \( \varphi(x, y) = y \), this implies by strategy-proofness \( \delta_e(x, y) \leq \delta_e(x, \hat{x}) \). This contradicts (4). Consequently, \( \varphi(z, y) = y \).

---

**Lemma 7**: Let \( y, v, w \in \mathbb{R}^m \) with \( \varphi(v, y) = v \) and \( \varphi(w, y) = w \). Let \( x \in [v, w[ \). Then \( \varphi(x, y) \notin [y, x[ \).

**Proof**: Let \( u = \varphi(x, y) \). Suppose \( u \in [y, x[ \). Without loss of generality assume \( \angle(v, x, u) \geq 90^\circ \) (otherwise interchange \( v \) and \( w \)). Then we can take \( z \in \)
[y, x, ∞]\(\[y, x]\) such that \(\delta_g(z, v) < \delta_g(z, u)\). By Lemma 1, \(\varphi(x, u) = u\). Hence, by Lemma 6, \(\varphi(z, \varphi) = u\). Let \(a = \varphi(z, y)\). Then by strategy-proofness, \(\delta_g(y, a) \leq \delta_g(y, u)\) and \(\delta_g(z, a) \leq \delta_g(z, v) < \delta_g(z, u)\). So, \(\delta_g(y, a) + \delta_g(a, z) < \delta_g(y, u) + \delta_g(z, u) = \delta_g(y, z)\). This contradicts the triangular inequality for metrics. Consequently, \(\varphi(x, y) \neq [y, x]\).

The next two lemmas establish weak continuity properties of \(\varphi\).

**Lemma 8**: Let \(x, x', y \in \mathbb{R}^m (i \in \mathbb{N})\) with \(x' \rightarrow x\) and \(\varphi(x, y) = y\). Then \(\varphi(x', y) \rightarrow y\).

**Proof**: In view of Lemma 3 we may assume without loss of generality that \(\varphi(x', y)\) converges, say to \(w\). By strategy-proofness it follows easily that

\[
\delta_g(x, w) = \delta_g(x, y) .
\]

Also, by strategy-proofness, \(\delta_g(w, \varphi(w, y)) \leq \lim_{i \to \infty} \delta_g(w, \varphi(x', y)) = 0\). Hence,

\[
\varphi(w, y) = w .
\]

Suppose \(w \neq y\), and let \(\hat{x} = \frac{1}{2}y + \frac{1}{2}w\). Then by (5) we have, \(\delta_g(x, \hat{x}) < \delta_g(x, y)\). Hence, by strategy-proofness,

\[
\varphi(\hat{x}, y) \neq \hat{x} .
\]

On the other hand, by Lemma 1, \(\varphi(y, y) = y\). Hence by (6) and Lemma 4, \(\varphi(\hat{x}, y) = \hat{x}\), which contradicts (7). Consequently, \(w = y\), so \(\varphi(x', y) \rightarrow y\).

**Lemma 9**: Let \(x, x', w \in \mathbb{R}^m (i \in \mathbb{N})\) with \(x' \rightarrow x\) and \(\varphi(x', w) \rightarrow w\). Then \(\varphi(x, w) = w\).

**Proof**: By strategy-proofness, \(\delta_g(w, \varphi(w, w)) \leq \lim_{i \to \infty} \delta_g(w, \varphi(x', w)) = 0\). Hence, \(\varphi(w, w) = w\). Let \(u = \varphi(x, w)\). Suppose \(u \neq w\). Then by Lemma 5, \(\delta_g(x, u) < \delta_g(x, w)\). So for \(i\) large enough, \(2\delta_g(x, x') + \delta_g(w, \varphi(x', w)) < \delta_g(x, w) - \delta_g(x, u)\), and therefore \(\delta_g(x', u) \leq \delta_g(x', x) + \delta_g(x, u) < -\delta_g(x', x) - \delta_g(w, \varphi(x', w)) + \delta_g(x, w) \leq \delta_g(x', \varphi(x', w))\). This contradicts strategy-proofness.

The following (rather technical) lemma will be used to establish continuity of \(\varphi\) for the case \(m = n = 2\) in Lemma 11.
Lemma 10: If \( \varphi \) is not continuous, then there are \( x, y, v, w \in \mathbb{R}^m \), with \( v = \varphi(x, y) \), \( w = \varphi(w, y) = \varphi(x, w) \neq v \), \( \delta_g(x, v) = \delta_g(x, w) \) and \( \delta_g(y, v) = \delta_g(y, w) \).

Proof: Suppose \( \varphi \) is not continuous in the first coordinate. Then there are \( x, x^i, y \in \mathbb{R}^m \) \( (i \in \mathbb{N}) \) with \( x^i \to x \) and

\[
\varphi(x^i, y) \to \varphi(x, y). \tag{8}
\]

Let

\[
v = \varphi(x, y). \tag{9}
\]

In view of Lemma 3 we may assume that \( \varphi(x^i, y) \) converges, say to \( w \). By (8) and (9),

\[
v \neq w. \tag{10}
\]

By strategy-proofness it is easily established that

\[
\varphi(w, y) = w. \tag{11}
\]

and

\[
\delta_g(x, v) = \delta_g(x, w). \tag{12}
\]

By Lemma 1, \( \varphi(x, v) = v \). So by Lemma 8, \( \varphi(x^i, v) \to v \). Hence by strategy-proofness,

\[
\delta_g(y, w) = \delta_g\left(y, \lim_{i \to \infty} \varphi(x^i, y)\right)
\]

\[
= \lim_{i \to \infty} \delta_g(y, \varphi(x^i, y))
\]

\[
\leq \lim_{i \to \infty} \delta_g(y, \varphi(x^i, v))
\]

\[
\leq \delta_g\left(y, \lim_{i \to \infty} \varphi(x^i, v)\right) = \delta_g(y, v). \tag{13}
\]
Also by strategy-proofness, $\delta_E(w, \varphi(x^i, w)) \leq \delta_E(w, \varphi(x^j, y))$ for all $i \in \mathbb{N}$, so $\varphi(x^i, w) \rightarrow w$. Then by Lemma 9,

$$\varphi(x, y) = w.$$  (14)

Hence by strategy-proofness, $\delta_E(y, v) \leq \delta_E(y, w)$. So by (13),

$$\delta_E(y, v) = \delta_E(y, w).$$  (15)

Now (9), (10), (11), (12), (14), and (15) give the desired result.

**Lemma 11:** Let $m = 2$. Then $\varphi$ is continuous.

**Proof:** Suppose $\varphi$ is not continuous. Then Lemma 10 implies that there are $x, y, v, w \in \mathbb{R}^m$, with $v = \varphi(x, y), w = \varphi(w, y) = \varphi(x, w) \neq v$,

$$\delta_E(x, v) = \delta_E(x, w)$$  (16)

and

$$\delta_E(y, v) = \delta_E(y, w).$$  (17)

Let $d = \frac{1}{2}v + \frac{1}{2}w$. (See Figure 1.) Let $u = \varphi(d, y)$. By (16),

$$\delta_E(x, d) < \delta_E(x, v).$$  (18)

So by strategy-proofness, $u \neq d$. Hence by Lemma 7, $u \notin [y, d]$. By Lemma 1 and range convexity, $d = \varphi(d, d)$. So by Lemma 5, $\angle(y, u, d) \geq 90^\circ$. Hence,

$$\delta_E(y, u) < \delta_E(y, d).$$  (19)

Without loss of generality assume that $\angle(w, d, u) < 90^\circ$ (otherwise interchange the roles of $v$ and $w$). Then we can take $z \in [u, d, \infty \setminus [u, d]$ with $\angle(w, v, z) = 90^\circ$. Let $a = \varphi(z, y)$. Then by strategy-proofness, $\delta_E(z, a) \leq \delta_E(z, v)$. Therefore, $[y, a]$ has nonempty intersection with the straight line through $v$ and $w$.

By Lemma 1, $\varphi(d, u) = u$. So Lemma 6 implies that $\varphi(z, u) = u$. By strategy-proofness and (19), $\delta_E(y, a) \leq \delta_E(y, u) < \delta_E(y, d)$. Therefore, by (18), $[y, a]$ has
empty intersection with the straight line through \( v \) and \( w \), contradicting our earlier finding. Consequently, \( \varphi \) is continuous.

Lemma 12: Let \( m = 2 \). Let \( v, w, y \in \mathbb{R}^2 \) with \( \varphi(v, y) = v \) and \( \varphi(w, y) = w \). Then \( \varphi(x, y) = x \) for all \( x \in [v, w] \).

Proof: Let \( \psi: \mathbb{R}^2 \to \mathbb{R}^2 \) be defined by \( \psi(x) = \varphi(x, y) \) for \( x \in \mathbb{R}^2 \). It is straightforward to verify that, since \( \varphi \) is strategy-proof with respect to \( \delta_p \), the one-individual voting scheme \( \psi \) is strategy-proof with respect to \( \delta_p \). By Lemma 11, \( \varphi \) is continuous. So \( \psi \) is continuous. Hence by Theorem 4, \( \psi \) is range convex. Let \( x \in [v, w] \). Then by range convexity, \( x \) is in the range of \( \psi \). So \( x = \psi(u) \) for some \( u \in \mathbb{R}^2 \). Therefore, by Lemma 1, \( \psi(x) = x \). Consequently, \( \varphi(x, y) = x \).

5 n Individuals

This section contains our main results, which concern the case where the number \( n \) of individuals is arbitrary, but the dimension \( m \) is equal to two.

The groundwork for our results was laid in the two preceding sections. Here, we need one more technical lemma, which extends Lemma 12 to \( n \) individuals.
Range Convexity, Continuity, and Strategy-Proofness of Voting Schemes

In the sequel sometimes a notation like $x^S (x \in \mathbb{R}^n, S \subseteq N)$ will be used in a given profile; this means that to every individual in $S$ the point $x$ is assigned by that profile.

Lemma 13: Let $\varphi : (\mathbb{R}^2)^N \rightarrow \mathbb{R}^2$ satisfy range convexity and strategy-proofness with respect to $\delta_k$. Let $k \in N$, $p \in (\mathbb{R}^2)^{N\setminus\{k\}}$. Let $v, w \in \mathbb{R}^2$ with $\varphi(v^{k}, p) = v$ and $\varphi(w^{k}, p) = w$. Then $\varphi(x^{k}, p) = x$ for all $x \in [v, w]$.

Proof: Without loss of generality assume $k = 1$. Let $S_j = \{1, \ldots, j\}$ for $j \in N$. For every $j \in N$ let $p' \in (\mathbb{R}^2)^{N\setminus S_j}$ be such that $p'$ coincides with $p$ on $N\setminus S_j$. Let $i$ be the smallest element of $N$ such that for all $\tilde{v}, \tilde{w}, \tilde{x} \in \mathbb{R}^n$ with $\varphi(\tilde{v}^S, p') = \tilde{v}$, $\varphi(\tilde{w}^S, p') = \tilde{w}$ and $\tilde{x} \in [\tilde{v}, \tilde{w}]$ we have that $\varphi(\tilde{x}^S, p') = \tilde{x}$. Since the range convexity of $\varphi$ and Lemma 1 imply that the statement holds for $i = n$, a smallest $i$ for which it holds certainly exists.

Suppose $i > 1$, and let $\psi : (\mathbb{R}^2)^2 \rightarrow \mathbb{R}^2$ be defined by $\psi(x, y) = \varphi(x^{S_{i-1}}, y^{i-1}, p')$. Then, by repeated application of strategy-proofness of $\varphi$, it is not hard to verify that also $\psi$ satisfies strategy-proofness with respect to $\delta_k$. By the definition of $i$, $\psi$ also satisfies range convexity. So Lemma 12 implies that for all $\tilde{v}, \tilde{w}, \tilde{x} \in \mathbb{R}^2$ with $\psi(\tilde{v}, p(i)) = \tilde{v}$, $\psi(\tilde{w}, p(i)) = \tilde{w}$ and $\tilde{x} \in [\tilde{v}, \tilde{w}]$ we have that $\psi(\tilde{x}, p(i)) = \tilde{x}$. Hence, for all $\tilde{v}, \tilde{w}, \tilde{x} \in \mathbb{R}^2$ with $\varphi(\tilde{x}^{S_{i-1}}, p^{i-1}) = \tilde{x}$, $\varphi(\tilde{v}^{S_{i-1}}, p^{i-1}) = \tilde{v}$ and $\tilde{x} \in [\tilde{v}, \tilde{w}]$ we have that $\varphi(\tilde{x}^{S_{i-1}}, p^{i-1}) = \tilde{x}$. This contradicts the definition of $i$.

Consequently, $i = 1$, from which the lemma follows. \(\square\)

The following theorem extends Theorem 4.

Theorem 5: Let $\varphi : (\mathbb{R}^2)^N \rightarrow \mathbb{R}^2$ satisfy strategy-proofness with respect to $\delta_k$. Then $\varphi$ is range convex if, and only if, $\varphi$ is continuous.

Proof:

i) Let $\varphi$ be range convex. Let $k \in N$ and let $p \in (\mathbb{R}^2)^{N\setminus\{k\}}$. Let $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $\psi(x) = \varphi(x^{k}, p)$ for all $x \in \mathbb{R}^2$. Then by Lemma 13, $\psi$ is range convex. Hence by Theorem 4, $\psi$ is continuous. Since $p$ was arbitrary, this implies that $\varphi$ is continuous in the $k$-th component. Since $k$ was arbitrary, this implies that $\varphi$ is continuous.

ii) Let $\varphi$ be continuous. Let $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $\psi(x) = \varphi(x^N)$ for all $x \in \mathbb{R}^2$. Then $\psi$ is continuous. Also, $\psi$ satisfies strategy-proofness with respect to $\delta_k$. Hence by Theorem 4, $\psi$ is range convex. Lemma 1 implies that the range of $\varphi$ coincides with the range of $\psi$. Consequently, $\varphi$ is range convex. \(\square\)

The following corollary is an immediate consequence of Theorems 1 and 5.
Corollary 1: Let \( n \geq 2 \). Then \( \varphi : \mathbb{R}^2 \to \mathbb{R}^2 \) is anonymous, range convex, and strategy-proof with respect to \( \delta_b \) if, and only if, \( \varphi \) is a coordinatewise median voting scheme with \( n + 1 \) constant points.

If range convexity is replaced by surjectivity we obtain the following result.

Corollary 2: Let \( n \geq 2 \). Then \( \varphi : \mathbb{R}^2 \to \mathbb{R}^2 \) is anonymous, surjective, and strategy-proof with respect to \( \delta_b \) if, and only if, \( \varphi \) is a coordinatewise median voting scheme with \( n - 1 \) constant points.

**Proof:**

i) Let \( \varphi : \mathbb{R}^2 \to \mathbb{R}^2 \) satisfy anonymity, surjectivity, and strategy-proofness with respect to \( \delta_b \). By surjectivity the range of \( \varphi \) coincides with \( \mathbb{R}^2 \). So \( \varphi \) is range convex. Hence Corollary 1 implies that \( \varphi \) is a coordinatewise median voting scheme with \( n + 1 \) constant points. Surjectivity of \( \varphi \) implies that in each coordinate the maximum of the constant points must be equal to \( \infty \) and the minimum to \(-\infty\). Since these values can be cancelled out coordinatewise, it follows that \( \varphi \) is a coordinatewise median voting scheme with \( n - 1 \) constant points.

ii) Let \( \varphi \) be a coordinatewise median voting scheme with \( n - 1 \) constant points. Then \( \varphi(x^N) = x \) for every \( x \in \mathbb{R}^n \): \( \varphi \) is surjective. The other properties of \( \varphi \) follow from the fact that \( \varphi \) is a coordinatewise median voting scheme with \( n + 1 \) constant points, and Theorem 1.

Since by Lemma 1 for a strategy-proof voting scheme surjectivity and unanimity are equivalent, our final result is an immediate consequence of Corollary 2.

Corollary 3: Let \( n \geq 2 \). Then \( \varphi : \mathbb{R}^2 \to \mathbb{R}^2 \) is anonymous, unanimous and strategy-proof with respect to \( \delta_b \) if, and only if, \( \varphi \) is a coordinatewise median voting scheme with \( n - 1 \) constant points.

**References**

Range Convexity, Continuity, and Strategy-Proofness of Voting Schemes