Coalitional Strategy-Proofness in Economies with Single-Dipped Preferences and the Assignment of an Indivisible Object

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We study two allocation models. In the first model, we consider the problem of allocating an infinitely divisible commodity among agents with single-dipped preferences. In the second model, a degenerate case of the first one, we study the allocation of an indivisible object to a group of agents. Our main result is the characterization of the class of Pareto optimal and coalitionally strategy-proof allocation rules. Alternatively, this class of rules, which largely consists of serially dictatorial components, can be characterized by Pareto optimality, strategy-proofness, and weak non-bossiness (in terms of welfare). Furthermore, we study properties of fairness such as anonymity and no-envy. Journal of Economic Literature Classification Numbers: D63, D71. © 2001 Academic Press

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1. INTRODUCTION

In many economic and political situations the preferences of the agents involved are private information. Depending on the decision mechanism to be used, agents may have an incentive to strategically misrepresent their preferences in order to influence the final outcome to their own advantage, probably at the expense of others. Misrepresentation of preferences can be avoided by imposing strategy-proofness on the mechanism. A decision mechanism is strategy-proof if no agent ever can gain by misrepresenting his preferences, irrespective of the preferences announced by the other agents.

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We consider problems of allocating a private commodity among a group of agents. The literature on strategy-proofness for private goods economies, similar to the general literature on strategy-proofness (Gibbard (1973) and Satterthwaite (1975)), starts off with an impossibility result. For two-agent two-goods exchange economies, Hurwicz (1972) showed that even for a restricted domain that contains all translated Cobb–Douglas preferences, essentially no allocation mechanism satisfies strategy-proofness and Pareto optimality. This incompatibility does not occur on certain restricted domains; the domain of single-peaked preferences is a well-known example.

In this paper we consider two allocation models. In the first model we deal with the problem of allocating some perfectly divisible commodity among a group of agents with so-called “single-dipped” preferences: preferences are single-dipped if alternatives can be ordered in such a way that every agent has a single worst alternative, his dip, and his welfare increases monotonically in either direction away from his dip. For example, consider two types of work which have negative cross-effects like, perhaps, teaching and management in a university: combinations of the two types of work may be less preferred than pure one-type tasks.

Other examples can be derived from two-goods exchange economies with either fixed prices and strictly quasi-convex utility functions or non-convex budget sets and classical economic preferences. In both situations, induced single-dipped preferences on the budget boundary result. The domain of single-dipped preferences in a public goods context was first considered by Vickrey (1960). Inada (1964) studies single-dipped preferences over triples of alternatives and Peremans and Storcken (1997) consider the problem of locating a public facility with strongly negative externalities which induce single-dipped preferences on the set of admissible locations.

In the second model, we consider the allocation of an indivisible object, for instance a job or a prize, among a group of agents. Since there are only two possible outcomes for each agent, individual preferences can be interpreted as degenerate single-dipped preferences. Hence, the problem of assigning an indivisible object to a set of agents is embedded in the allocation problem with single-dipped preferences. Most results we derive for the allocation of an infinitely divisible commodity are easily “translated” to the assignment problem.

Our main result (for both models) is a characterization of the class of rules that satisfy Pareto optimality and coalitional strategy-proofness (see Theorem 1 and Corollary 2): the main characteristic of any of those rules is that the whole amount of the commodity is assigned to a single agent and the rule consists for the largest part of serially dictatorial components. An alternative characterization is obtained by weakening coalitional
strategy-proofness to strategy-proofness plus a weak non-bossiness condition\(^1\) (see Theorem 1 and Corollary 2). Furthermore, it turns out that one of our most important properties, namely Pareto optimality, is not compatible with properties of fairness such as anonymity and no-envy.

Klaus et al. (1997) and Klaus (1998) provide similar characterizations involving solidarity properties (replacement-domination and population-monotonicity) and consistency properties (consistency and separability). Recently, Ehlers (1998) extended the model we discuss here by allowing for probabilistic allocation rules. It is noteworthy that in a “probabilistic world” some of the incompatibilities we establish do not occur.

In a recent paper, Pápai (1998) characterizes the set of serial dictatorship rules for an assignment model that is similar to the one we study. On the domain of strict preferences, Pápai (1998) independently derives two characterizations of serial dictatorships that correspond to our Corollary 2. Apart from an alternative proof technique, our results (Corollary 2) show the effect of dropping the standard requirement that preferences are strict in this kind of assignment situation.

The paper is organized as follows. In Section 2, we introduce the allocation model with single-dipped preferences and the main properties Pareto optimality and (coalitional) strategy-proofness. In Section 3, we discuss compatibilities and incompatibilities of Pareto optimality and (coalitional) strategy-proofness with other well-known properties. The main compatibility result is the characterization of all Pareto optimal and coalitional strategy-proof rules (Theorem 1). In Section 4, we show how compatibility and incompatibility results derived in Section 3 extend to the allocation of an indivisible object.

2. THE MODEL

Let \( \Omega \in \mathbb{R}_+ \) be the amount of an infinitely divisible commodity, or the (social) endowment, that has to be distributed among a non-empty and finite set \( N \) of agents. Each agent \( i \in N \) is equipped with a “single-dipped” preference relation \( R_i \) defined over the closed interval \([0, \Omega]\). Single-dippedness of \( R_i \) means that there exists a point \( d(R_i) \in [0, \Omega] \), called the dip of agent \( i \), with the following property: for all \( x, y \in [0, \Omega] \) with \( x < y \leq d(R_i) \) or \( x > y \geq d(R_i) \) we have \( xP_i y \). As usual, \( xR_i y \) is interpreted as “\( x \) is weakly preferred to \( y \),” and \( xP_i y \) as “\( x \) is strictly preferred to \( y \).” Furthermore, \( xI_i y \) means that agent \( i \) is indifferent between \( x \) and \( y \). Without loss of

\(^1\)A rule is weakly non-bossy (in terms of welfare), if no agent can influence some other agent’s welfare without affecting his own allotment.
generality, we assume that the agents’ preference relations are continuous as well.

By $\mathcal{D}$ we denote the class of all continuous and single-dipped preference relations over $[0, \Omega]$. Let $R = (R_i)_{i \in N} \in \mathcal{D}^N$ be a (preference) profile. Since the set of agents and the endowment are fixed we simply designate an economy by a profile of preference relations. Thus, the class of all economies equals $\mathcal{D}^N$.

Let $X = \{\chi^i\}_{i \in N}$ denote the set of unit vectors in $\mathbb{R}^N$, where $\chi^i_j = 1$ if $j = i$ and $\chi^i_j = 0$ if $j \neq i$. Let $\Omega X = \{\Omega \chi^i\}_{i \in N}$. Then, a feasible allocation for $R \in \mathcal{D}^N$ is a nonnegative point $x \in \mathbb{R}^N_+$ such that $\sum_i x_i = \Omega$. An allocation rule $\varphi$, or a rule for short, is a function that assigns to every $R \in \mathcal{D}^N$ a feasible allocation, denoted $\varphi(R)$. Given $i \in N$, we call $\varphi_i(R)$ the allotment of agent $i$.

We are interested in rules that assign Pareto optimal allocations: an allocation assigned by the rule cannot be changed in a way that no agent is worse off and some agent is better off.

Pareto Optimality. For all $R \in \mathcal{D}^N$, there is no feasible allocation $x \in \mathbb{R}^N_+$ such that for all $i \in N$, $x_i R_i \varphi_i(R)$, and for some $j \in N$, $x_j P_j \varphi_j(R)$.

A weaker notion of Pareto optimality is the following weak Pareto optimality: an allocation assigned by the rule cannot be changed in a way that all agents are better off.

Weak Pareto Optimality. For all $R \in \mathcal{D}^N$, there is no feasible allocation $x \in \mathbb{R}^N_+$ such that for all $i \in N$, $x_i P_i \varphi_i(R)$.

First, we present a simple description of Pareto optimality for our model. For this description we introduce the following “partition” of the agents. For an economy $R \in \mathcal{D}^N$ denote the set of agents who strictly prefer $\Omega$ to 0 by $N_{\Omega}(R) = \{i \in N \mid \Omega P_i 0\}$, the set of agents who are indifferent between 0 and $\Omega$ by $N_{0, \Omega}(R) = \{i \in N \mid 0I_i \Omega\}$, and the set of agents who strictly prefer 0 to $\Omega$ by $N_0(R) = \{i \in N \mid 0P_i \Omega\}$.

Hence, for all $R \in \mathcal{D}^N$, $N = N_{\Omega}(R) \cup N_{0, \Omega}(R) \cup N_0(R)$ and the sets $N_{\Omega}(R)$, $N_{0, \Omega}(R)$, and $N_0(R)$ are pairwise disjoint. However, since some of the sets $N_{\Omega}(R)$, $N_{0, \Omega}(R)$, and $N_0(R)$ are possibly empty, strictly speaking, $\{N_{\Omega}(R), N_{0, \Omega}(R), N_0(R)\}$ may not constitute a partition.

Lemma 1. A rule $\varphi$ is Pareto optimal if and only if for all $R \in \mathcal{D}^N$ the following holds.

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3In Klaus (1998) we consider variations of the population as well as variations of the endowment.
FIG. 1. Pareto optimal shares as described in Lemma 1. (a) Lemma 1, Case 1: $N/\Omega_1(R) \neq \emptyset$. Necessary conditions for Pareto optimality of allotments for agents $i, j \in N/\Omega_1(R)$, are: $x_i \in (0) \cup (a, \Omega)$ and $x_j \in [0, \Omega]$. (b) Lemma 1, Case 3: $N_0(R) = N$. Necessary conditions for Pareto optimality of allotments for agents $k, l \in N_0(R)$, are: $y_k \in (0, b) \cup \Omega$ and $y_l \in (0, \Omega]$.

Case 1. If $N_{i_1}(R) \neq \emptyset$, then
- for all $i \notin N_{i_1}(R)$, $\varphi_i(R) = 0$ and
- for all $i \in N_{i_1}(R)$, either $\varphi_i(R) = 0$ or $\varphi_i(R)Pi_{0.3}$.

Case 2. If $N_{i_2}(R) = \emptyset$ and $N_{0, i_2}(R) \neq \emptyset$, then
- for some $j \in N_{0, i_2}(R)$, $\varphi(R) = \Omega \chi^j$.  

Case 3. If $N_0(R) = N$, then
- for all $i \in N$, either $\varphi_i(R) = \Omega$ or $\varphi_i(R)Pi_{0.4}$.

Proof. See Appendix.

Next, in addition to Pareto optimality, we are interested in strategy-proofness. Strategy-proofness states that no agent ever benefits from misrepresenting his preferences. In game theoretical terms, an allocation rule is strategy-proof if in its associated direct revelation game form, it is a weakly dominant strategy for each agent to announce his true preference relation. Before we give the formal definition, we introduce some notation.

For $R \in \mathcal{D}^N$, $i \in N$, and $\bar{R}_i \in \mathcal{D}$, $(\bar{R}_i, R_{-i})$ denotes the profile obtained from $R$ by replacing $R_i$ by $\bar{R}_i$. We call $\bar{R} = (\bar{R}_i, R_{-i})$ an $i$-deviation from $R$. 

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3See Fig. 1a.
4See Fig. 1b.
Strategy-Proofness. For all $R$, $\bar{R} \in \mathcal{D}^N$ and all $j \in N$, if $\bar{R}$ is a $j$-deviation from $R$, then $\varphi_j(R)R_j\varphi_j(\bar{R})$.

A strengthening of strategy-proofness is the following coalitional strategy-proofness: no group, or coalition, of agents can ever benefit from misrepresenting their preferences.

Let $M \subseteq N$. For $R \in \mathcal{D}^N$ the restriction $(R_i)_{i \in M} \in \mathcal{D}^M$ of $R$ to $M$ is denoted by $R_M$. Let $C \subseteq N$. Then, $N \setminus C = \{i \in N \mid i \notin C\}$.

Coalitional Strategy-Proofness. For all $R \in \mathcal{D}^N$ and all $C \subseteq N$ there exists no $\bar{R} \in \mathcal{D}^N$ with $\bar{R}_{N \setminus C} = R_{N \setminus C}$ such that for all $i \in C$, $\varphi_i(R)R_i\varphi_i(R)$ and for some $j \in C$, $\varphi_j(R)P_j\varphi_j(R)$.

3. COMPATIBILITIES AND INCOMPATIBILITIES

In this section, we study the compatibility of Pareto optimality and strategy-proofness with other well-known properties for rules. The following example shows that the class of Pareto optimal and strategy-proof rules also contains "erratic" rules, i.e., rules that rely on information that we do not consider as crucial for the problem at hand.

Example 1. In this example we assign the whole endowment $\Omega$ to a single agent. The choice of the agent who receives $\Omega$ depends on whether the dip of agent 1 is a rational or an irrational number. The following rule $\tilde{\varphi}$ satisfies Pareto optimality and strategy-proofness. Without loss of generality, we define $\tilde{\varphi}(R)$ for $N = \{1, 2, 3\}$. Let $R \in \mathcal{D}^N$.

Case 1. $N_{\Omega}(R) \neq \emptyset$. If $N_{\Omega}(R) = \{i\}$, then $\tilde{\varphi}(R) = \Omega \chi^i$. If $|N_{\Omega}(R)| > 1$, then

$$\tilde{\varphi}(R) = \begin{cases} \Omega \chi^1 & \text{if } \Omega \in \mathbb{Q} \\ \Omega \chi^2 & \text{if } 0 \not\in \mathbb{Q} \\ \Omega \chi^3 & \text{if } d(R_1) \not\in \mathbb{Q} \end{cases} \quad (1)$$

Case 2. $N_{\Omega}(R) = \emptyset$ and $N_{0, \Omega}(R) \neq \emptyset$. Then $\tilde{\varphi}(R) = \Omega \chi^i$ for some $i \in N_{0, \Omega}(R)$.

Case 3. $N = N_{\emptyset}(R)$. Then

$$\tilde{\varphi}(R) = \begin{cases} \Omega \chi^2 & \text{if } d(R_1) \in \mathbb{Q} \\ \Omega \chi^3 & \text{if } d(R_1) \not\in \mathbb{Q} \end{cases} \quad (2)$$

♦
For an example of a rule that satisfies Pareto optimality and strategy-proofness but that does not always assign the whole endowment to a single agent, we refer to Example 6.

In the remainder of this section, we add various additional properties and study how this narrows down the class of Pareto optimal and strategy-proof rules.

Non-Bossiness and Coalitional Strategy-Proofness

In Example 1 we described a rule that in some situations assigns the whole endowment in a so-called “bossy” way, i.e., by unilaterally changing his preference relation, one of the agents can influence the allotments of the other agents even though his allotment remains the same. Hence, he can manipulate the allotments of the remaining agents without changing his own allotment. A rule that does not allow for this kind of manipulation is called non-bossy. This concept of (non-)bossiness was introduced by Satterthwaite and Sonnenschein (1981).

We think that in an allocation model that allows agents to be indifferent among different allotments, this original notion of non-bossiness is too strong because even though the allotments of some agents might change, it might be the case that all allotments remain the same in terms of welfare. We therefore consider two weaker versions of the original property of non-bossiness. First we consider non-bossiness in terms of welfare: no agent can influence some other agent’s welfare without affecting his own welfare.

Non-bossiness (in terms of welfare). For all $R, \bar{R} \in \mathcal{D}^N$ and all $j \in N$, if $\bar{R}$ is a $j$-deviation from $R$ and $\varphi_j(R)I_j\varphi_j(\bar{R})$, then for all $i \in N \setminus \{j\}$, $\varphi_i(R)I_i\varphi_i(\bar{R})$.

In the sequel we use the shorter phrase of non-bossiness for non-bossiness in terms of welfare.

As we will see later (Example 2), Pareto optimality, strategy-proofness, and non-bossiness are incompatible. However, the following slightly weaker version of non-bossiness turns out to be compatible with Pareto optimality and strategy-proofness. Weak non-bossiness in terms of welfare states that no agent can influence some other agent’s welfare without affecting his own allotment.

Weak non-bossiness (in terms of welfare). For all $R, \bar{R} \in \mathcal{D}^N$ and all $j \in N$, if $\bar{R}$ is a $j$-deviation from $R$ and $\varphi_j(R)I_j\varphi_j(\bar{R})$, then for all $i \in N \setminus \{j\}$, $\varphi_i(R)I_i\varphi_i(\bar{R})$.

In the sequel we use the shorter phrase of weak non-bossiness for weak non-bossiness in terms of welfare.

Next, we not only show that the class of rules satisfying Pareto optimality, strategy-proofness, and weak non-bossness equals the class of Pareto optimal
and *coalitionally strategy-proof* rules, but we also characterize the whole class of rules. In order to satisfy all properties mentioned above, the whole endowment is assigned to a single agent. The selection of this agent can be described as follows.

We fix a linear order of the set of agents and ask the agent who is first in this order whether he prefers $\Omega$ to 0. If he prefers the whole endowment, he receives it and we are done. If not, we ask the second agent the same question, and so on. Hence, the first agent according to the fixed order who prefers the whole endowment to nothing receives it. If no agent strictly prefers $\Omega$ to 0 and at least one agent is indifferent between 0 and $\Omega$, then the endowment can be assigned “arbitrarily” to any of the indifferent agents. If all agents strictly prefer 0 to $\Omega$, there exists a preselected agent who receives $\Omega$. Loosely speaking, there is a “scapegoat” who always receives the whole endowment when nobody else wants it.

For a formal description, we introduce additional notation.

A permutation $\pi$ on $N$ is a bijective function $\pi : N \rightarrow N$. By $\Pi^N$ we denote the set of all permutations on $N$. Since for $\pi \in \Pi^N$ and $i \in N$, $\pi(i)$ can also be interpreted as the position of $i$ in a linear order on $N$, we alternatively refer to $\pi$ as the linear order on $N$. By $G^N$ we denote the set of choice functions $g : \mathcal{D}^N \rightarrow N$ such that $g(R) \in N_{0,\Omega}(R)$ if $N_{0,\Omega}(R) \neq \emptyset$.

**Selection $f_{\pi,g,k}$.** Let $\pi \in \Pi^N$, $g \in G^N$, and $k \in N$. Then the selection $f_{\pi,g,k} : \mathcal{D}^N \rightarrow N$ (based on $\pi$, $g$, and $k$) is defined as follows. Let $R \in \mathcal{D}^N$.

**Case 1.** If $N_{\Omega}(R) \neq \emptyset$, then $f_{\pi,g,k}(R) = \arg\min \{\pi(i) \mid i \in N_{\Omega}(R)\}$.

**Case 2.** If $N_{\Omega}(R) = \emptyset$ and $N_{0,\Omega}(R) \neq \emptyset$, then $f_{\pi,g,k}(R) = g(R)$.

**Case 3.** If $N = N_{0}(R)$, then $f_{\pi,g,k}(R) = k$.

Note that in Case 1 the selection rule equals a serial-dictatorship. Also Case 3 can be interpreted as (degenerated) serially dictatorial.

**Theorem 1.** Let $\varphi$ be a rule. Then the following statements are equivalent.

(i) $\varphi$ satisfies Pareto optimality and coalitional strategy-proofness.

(ii) $\varphi$ satisfies Pareto optimality, strategy-proofness, and weak non-bossiness.

(iii) There exist $\pi \in \Pi^N$, $g \in G^N$, and $k \in N$ such that for all $R \in \mathcal{D}^N$, $\varphi(R) = \Omega_f^{f_{\pi,g,k}(R)}$.
Since on the domain of two-agent economies each Pareto optimal rule satisfies weak non-bossiness, and coalitional strategy-proofness simply boils down to strategy-proofness, Theorem 1 implies the following.

**Corollary 1.** Let $|N| = 2$. Then a rule $\varphi$ satisfies Pareto optimality and strategy-proofness if and only if there exist $\pi \in \Pi^N$, $g \in G^N$, and $k \in N$ such that for all $R \in \mathcal{D}^N$,

$$\varphi(R) = \Omega_1^{f_{\pi,k}(R)}.$$  

(4)

We divide the proof of Theorem 1 in the following steps.

First, we show that coalitional strategy-proofness implies weak non-bossiness (Lemma 2). Then, we prove that a rule that satisfies Pareto optimality, strategy-proofness, and weak non-bossiness assigns the whole endowment to a single agent (Lemma 3). Next, we show that Pareto optimality, strategy-proofness, and weak non-bossiness imply coalitional strategy-proofness (Lemma 4). Finally, we finish the proof by identifying the class of Pareto optimal and coalitional strategy-proof (strategy-proof and weak non-bossy) rules with the class of “serial-dictatorship” rules as described in Theorem 1 (iii).

**Lemma 2.** Let $\varphi$ be a rule that satisfies coalitional strategy-proofness. Then $\varphi$ satisfies weak non-bossiness.

**Proof.** Let $\varphi$ be coalitionally strategy-proof. Let $R \in \mathcal{D}^N$ and consider a j-deviation $\overline{R}$ from $R$ such that $\varphi_j(R) = \varphi_j(\overline{R})$. Hence, $\varphi_j(R)I_j\varphi_j(\overline{R})$ and $\varphi_j(\overline{R})I_j\varphi_j(\overline{R})$. In order to prove weak non-bossiness we assume, by contradiction, that there exists an agent $k \neq j$ such that either (a) $\varphi_k(\overline{R})P_k\varphi_k(R)$ or (b) $\varphi_k(R)P_k\varphi_k(\overline{R})$.

(a) Then, $\overline{R}_{N \setminus \{j,k\}} = R_{N \setminus \{j,k\}}$, $\varphi_j(\overline{R})I_j\varphi_j(R)$, and $\varphi_k(\overline{R})P_k\varphi_k(R)$ contradict coalitional strategy-proofness.

(b) Note that $\varphi_k(R)P_k\varphi_k(\overline{R})$. Then, $R_{N \setminus \{j,k\}} = \overline{R}_{N \setminus \{j,k\}}$, $\varphi_j(R)I_j\varphi_j(\overline{R})$, and $\varphi_k(R)P_k\varphi_k(\overline{R})$ contradict coalitional strategy-proofness.

Next, we show that a rule $\varphi$ that satisfies Pareto optimality, strategy-proofness, and weak non-bossiness assigns the whole endowment to a single agent.

Let $\mathcal{B}_\varphi = \{R \in \mathcal{D}^N \mid \varphi(R) \notin \Omega X^N\}$ be the set of economies that yields a broken allocation under $\varphi$, i.e., a feasible allocation where no agent obtains the whole endowment.

**Lemma 3.** Let $\varphi$ be a rule that satisfies Pareto optimality, strategy-proofness, and weak non-bossiness. Then, $\mathcal{B}_\varphi = \emptyset$. 

Proof. Let \( \varphi \) satisfy the properties listed in the lemma and suppose by contradiction that \( \mathcal{B}_\varphi \neq \emptyset \).

Given \( R \in \mathcal{B}_\varphi \), let \( B_\varphi(R) \) denote the set of agents with broken allotments at \( \varphi(R) \), i.e., \( B_\varphi(R) = \{ i \in N \mid \varphi_i(R) \notin \{0, \Omega\} \} \). Since \( \mathcal{B}_\varphi \neq \emptyset \) and \( N \) is finite, there exists a profile \( R \in \mathcal{B}_\varphi \) with a minimal number of agents who have a broken allotment. So, let \( \bar{R} \in \mathcal{B}_\varphi \) be such that \( \varphi(R) \notin \Omega \times \Omega \) and for all \( \bar{R} \in \mathcal{B}_\varphi \), \( |B_\varphi(\bar{R})| \geq |B_\varphi(R)| \). Hence, \( B_\varphi(R) \neq \emptyset \) and, by Lemma 1, either (a) for all \( i \in B_\varphi(R) \), \( \varphi_i(R) \neq 0 \) or (b) for all \( i \in B_\varphi(R) \), \( \varphi_i(R) \neq \Omega \). Hence, either (a) \( B_\varphi(R) \subseteq N_1(R) \) or (b) \( B_\varphi(R) \subseteq N_0(R) \).

(a) \( B_\varphi(R) \subseteq N_1(R) \). First, we show that, without loss of generality, we can assume that \( B_\varphi(R) = N_1(R) \).

(a.1) Suppose that \( B_\varphi(R) \subseteq N_1(R) \). Let \( i \in N_1(R) \) and \( i \notin B_\varphi(R) \). So, \( \varphi_i(R) = 0 \).

Let \( R^1 \in \mathcal{B}_\varphi \) be an \( i \)-deviation from \( R \) such that \( R^1 \in N_0(R^1) \). Hence, by Lemma 1, \( \varphi_i(R^1) = 0 \). Since \( \varphi_i(R^1) = \varphi_i(R) = 0 \), by weak non-bossiness, for all \( j \in N \setminus \{i\} \), \( \varphi_j(R^1) \neq \varphi_j(R) \). Thus, by Lemma 1, for all \( j \in N \setminus \{i\} \), \( \varphi_j(R^1) = \varphi_j(R) \). Hence, \( B_\varphi(R^1) = B_\varphi(R) \). Since \( B_\varphi(R^1) \subseteq N_1(R^1) \) and \( |N_1(R^1)| = |N_1(R)| - 1 \).

Next, suppose that \( B_\varphi(R) \subseteq N_1(R^1) \). Let \( i \in N_1(R^1) \) and \( i \notin B_\varphi(R^1) \).

Let \( R^2 \in \mathcal{B}_\varphi \) be an \( i \)-deviation from \( R^1 \) such that \( R^2 \in N_0(R^2) \). Then, as before, it follows that \( B_\varphi(R^2) = B_\varphi(R^1) \). Since \( B_\varphi(R^2) \subseteq N_1(R^2) \) and \( |N_1(R^2)| = |N_1(R^1)| - 1 \).

Since the set of agents \( N \) is finite, repeating the preceding steps, we finally end up with a profile \( \hat{R} \in \mathcal{B}_\varphi \) such that \( B_\varphi(\hat{R}) = B_\varphi(R) \) and \( B_\varphi(\hat{R}) = N_1(\hat{R}) \). Hence, it is without loss of generality to assume \( B_\varphi(R) = N_1(R) \) in the remainder of the proof.

(a.2) Let \( B_\varphi(R) = N_1(R) \). Let \( i \in B_\varphi(R) = N_1(R) \). First, consider an \( i \)-deviation \( R^1 \) from \( R \) such that \( \Omega P^1_i \varphi(R) \). Note that \( N_1(R^1) = N_1(R) \).

By strategy-proofness for the \( i \)-deviation \( R^1 \) from \( R \), \( \varphi_i(R) \neq \varphi_i(R^1) \). Hence, since \( i \in N_1(R) \), \( \varphi_i(R^1) \leq \varphi_i(R) \). By \( N_1(R^1) = N_1(R) \) and Lemma 1, it follows that either \( \varphi_i(R^1) = 0 \) or \( \varphi_i(R^1) \neq 0 \). The latter can only be true if \( \varphi_i(R^1) > \varphi_i(R) \), which contradicts strategy-proofness. Hence, \( \varphi_i(R^1) = 0 \). Because the number of agents in \( B_\varphi(R) \) is minimal for \( R \in \mathcal{B}_\varphi \) and \( B_\varphi(R) = N_1(R) \), it follows that \( R^1 \notin \mathcal{B}_\varphi \). Hence, by Lemma 1, there exists \( j \in B_\varphi(R^1) \setminus \{i\} \) such that \( \varphi_j(R^1) = \Omega \chi_i^1 \).

Next, consider a \( j \)-deviation \( R^2 \) from \( R^1 \) such that \( \Omega P^2_j \varphi_j(R) \). Hence, \( N_1(R^2) = N_1(R^1) \) and strategy-proofness for the \( j \)-deviation \( R^2 \) from \( R^1 \) implies \( \varphi(R^2) = \Omega \chi_i^1 \).

Suppose there exists \( k \in B_\varphi(R) \setminus \{i, j\} \). Then, consider a \( k \)-deviation \( R^3 \) from \( R^2 \) such that \( \Omega P^3_k \varphi_k(R) \). Since \( \varphi_k(R^2) = 0 \), strategy-proofness for the \( k \)-deviation \( R^3 \) from \( R^2 \) implies \( \varphi_k(R^3) = 0 \). Thus, by weak non-
bossiness and Lemma 1, \( \varphi(R^3) = \Omega \chi^l \). Next, suppose that there exists \( l \in B_\varphi(R) \setminus \{i, j, k\} \). Then, as before, it follows that for any \( l \)-deviation \( R^4 \) from \( R^3 \) such that \( \Omega P^i_4 0 P^i_4 \varphi_i(R), \varphi(R^4) = \Omega \chi^l \), etc.

Hence, after \(|B_\varphi(R)| \) steps, we obtain a profile \( \tilde{R} \in \mathcal{D}^N \) such that \( N_\Omega(\tilde{R}) = B_\varphi(R) \), for all \( m \notin N_\Omega(\tilde{R}) \), \( \tilde{R}_m = R_m \), and for all \( m \in N_\Omega(\tilde{R}) \), \( \Omega P^m_\Omega 0 P^m_\Omega \varphi_m(R) \). Furthermore, \( \varphi(\tilde{R}) = \Omega \chi^l \), and in particular \( \varphi(\tilde{R}) = 0 \).

Since \( i \in B_\varphi(R) \) was arbitrarily chosen, we can construct profile \( \tilde{R} \) proceeding from any \( j \in B_\varphi(R) \). Hence, for all \( j \in B_\varphi(R) \), \( \varphi(\tilde{R}) = 0 \). Since \( N_\Omega(\tilde{R}) = B_\varphi(R) \) this contradicts Pareto optimality.

(b) \( B_\varphi(R) \subseteq N_\Omega(\tilde{R}) \). Let \( i \in B_\varphi(R) \) and consider an \( i \)-deviation \( R^1 \) from \( R \) such that \( 0 P^i_1 0 P^i_1 \varphi_i(R) \). By strategy-proofness for the \( i \)-deviation \( R^1 \) from \( R \), \( \varphi_i(R)R \varphi_i(R^1) \). Hence, since \( i \in N_\Omega(\tilde{R}) \), \( \varphi_i(R^1) \geq \varphi_i(R) \). By \( N_\Omega(R^1) = N_\Omega(R) = N \) and Lemma 1, it follows that either \( \varphi_i(R^1) = \Omega \) or \( \varphi_i(R^1)P^i_1 \Omega \). The latter can only be true if \( \varphi_i(R^1) < \varphi_i(R) \), which contradicts strategy-proofness. Hence, \( \varphi(R^1) = \Omega \chi^l \).

Next, let \( j \in B_\varphi(R) \setminus \{i\} \) and consider a \( j \)-deviation \( R^2 \) from \( R^1 \) such that \( 0 P^j_2 0 P^j_2 \varphi_j(R) \). Since, \( \varphi_j(R^1) = 0 \), strategy-proofness for the \( j \)-deviation \( R^2 \) from \( R_1 \) implies \( \varphi_j(R^2) = 0 \). Thus, by weak non-bossiness and Lemma 1, \( \varphi(R^2) = \Omega \chi^l \). Next, suppose that there exists \( k \in B_\varphi(R) \setminus \{i, j\} \). Then, as before, it follows that for any \( k \)-deviation \( R^3 \) from \( R^2 \) such that \( 0 P^k_3 0 P^k_3 \varphi_k(R) \), \( \varphi(R^3) = \Omega \chi^l \), etc.

Hence, after \(|B_\varphi(R)| \) steps, we obtain a profile \( \tilde{R} \in \mathcal{D}^N \) such that for all \( m \notin B_\varphi(R) \), \( \tilde{R}_m = R_m \), and for all \( m \in B_\varphi(R) \), \( 0 P^m_\Omega 0 P^m_\Omega \varphi_m(R) \). Furthermore, \( \varphi(\tilde{R}) = \Omega \chi^l \).

Since \( i \in B_\varphi(R) \) was arbitrarily chosen, we can construct profile \( \tilde{R} \) proceeding from any \( j \in B_\varphi(R) \). Hence, for all \( j \in B_\varphi(R) \setminus \{i\} \), \( \varphi(\tilde{R}) = \Omega \chi^l \).

This contradicts \( \varphi(\tilde{R}) = \Omega \chi^l \).

Using Lemma 3 we can easily show the incompatibility of non-bossiness with Pareto optimality and strategy-proofness.

**Example 2.** Let \( N = \{1, 2\}, \Omega = 1 \) and consider a Pareto optimal, strategy-proof, and non-bossy rule \( \varphi \). Let \( R = (R_1, R_2) \in \mathcal{D}^N \) be such that \( 0P^1_1 \) and \( 0P^2_1 \). By Lemma 3, \( \varphi(R) = \chi^k \), \( k \in N_\Omega(R) = N \). Assume, without loss of generality, that \( \varphi(R) = \chi^2 \). Now, consider a 1-deviation \( \tilde{R} \) from \( R \) such that \( \Omega^1 1 \). Then, by Pareto optimality, \( \varphi(R) = \chi^1 \). Hence, for the 1-deviation \( R \) from \( R, \varphi(R)I_\varphi(R) \). Then, by non-bossiness, \( \varphi(\tilde{R})I_2 \varphi_2(R) \). However, \( \varphi_2(R) = 1, \varphi_2(\tilde{R}) = 0 \), and \( \varphi_2(\tilde{R})P_2 \varphi_2(R) \). Since \( \tilde{R} = \tilde{R}_2 \), this is in contradiction to \( \varphi_2(\tilde{R})I_2 \varphi_2(R) \).

Next, we prove that Pareto optimality, strategy-proofness, and weak non-bossiness imply coalitional strategy-proofness.
LEMMA 4. Let \( \varphi \) be a rule that satisfies Pareto optimality, strategyproofness, and weak non-bossiness. Then \( \varphi \) satisfies coalitional strategy-proofness.

Proof. Let \( \varphi \) satisfy the properties listed in the lemma and suppose by contradiction that \( \varphi \) is not coalitionally strategy-proof. Then there exist \( R, \overline{R} \in \mathcal{N} \) with \( R|_N \cap C = R_N|_C \) and \( C \subseteq N \) such that for all \( i \in C \), \( \varphi_i(\overline{R})Rj\varphi_j(R) \) and for some \( j \in C \), \( \varphi_j(\overline{R})Pj\varphi_j(R) \). By strategy-proofness, \(|C| > 1 \). Let \( c = |C|, C = \{i_1, \ldots, i_c\} \), and \( j = i_c \).

Case 1. \( N_{i_c}(R) \neq \emptyset \).

By Lemma 3, there exists \( k \in N \) such that \( \varphi(R) = \Omega \chi^k \). Hence, \( j, k \in N_{i_c}(R), k \notin C \), and \( \varphi(R) = \Omega \chi^k \).

First, we show that it is without loss of generality to assume that for all \( l \in C \setminus \{j\} \), \( l \notin N_{i_c}(R) \) and \( l \notin N_{i_c}(\overline{R}) \). Let \( l \in N_{i_c}(R) \) and consider the \( l \)-deviation \( \overline{R} \) from \( R \) with \( l \notin N_{i_c}(\overline{R}) \). Then, by Lemma 3 and Pareto optimality, \( \varphi_i(\overline{R}) = \varphi_i(R) = 0 \). Hence, by weak non-bossiness, \( \varphi(R) = \varphi(\overline{R}) \). Similarly, for \( l \notin N_{i_c}(\overline{R}) \) and \( l \)-deviation \( \overline{R} \) from \( \overline{R} \) with \( l \notin N_{i_c}(\overline{R}) \), \( \varphi(\overline{R}) = \varphi(\overline{R}) \). Furthermore, for all \( i \in C \), \( \varphi_i(\overline{R})\overline{R}\varphi_j(\overline{R}) \) and \( \varphi_j(\overline{R})Pj\varphi_j(\overline{R}) \). Repetition of the preceding steps yields the desired result.

Consider the \( i_c \)-deviation \( \overline{R}^1 \) from \( R \) with \( R_{i_c}^1 = \overline{R}_{i_c} \). Then, by Lemma 3 and Pareto optimality, \( \varphi_{i_c}(R) = \varphi_{i_c}(R^1) = 0 \). Hence, by weak non-bossiness, \( \varphi(R) = \varphi(R^1) \). Next, consider the \( i_2 \)-deviation \( R^2 \) from \( R^1 \) with \( R_{i_2}^1 = \overline{R}_{i_2} \). Similarly as before, it follows that \( \varphi(R^1) = \varphi(R^2) \). Hence, \( \varphi(R^2) = \varphi(R) \). Applying this argument several times yields for the profile \( R^{i_{c-1}} \) such that \( R_{i_c}^{i_{c-1}} = \overline{R}_{i_c}^{i_{c-1}} \) and \( R_{i_2}^{i_{c-1}} = \overline{R}_{i_2}^{i_{c-1}} \), \( \varphi(R^{i_{c-1}}) = \varphi(\overline{R}^{i_{c-1}}) \). Hence, \( \varphi_j(R^{i_{c-1}}) = \varphi(\overline{R}^{i_{c-1}}) = 0 \). Note that \( \overline{R} \) is a \( j \)-deviation from \( R^{i_{c-1}} \). By Lemma 3 and strategy-proofness, \( \varphi_j(\overline{R}) = 0 \). This is a contradiction to \( \varphi(\overline{R}) = \Omega \chi^k \).

Case 2. \( N_{i_c}(R) = \emptyset \) and \( N_{0, i_c}(R) \neq \emptyset \).

For all \( i \in N \) and all \( x \in [0, \Omega] \), \( \varphi_i(\overline{R})Rix \). Hence, there exists no \( j \in N \) and no \( x \in [0, \Omega] \) such that \( x = \varphi_j(\overline{R})Pj(\overline{R}) \).

Case 3. \( N = N_{i_c}(\overline{R}) \).

Hence, \( \varphi(R) = \Omega \chi^k \) and there exists \( k \notin C \) such that \( \varphi(R) = \Omega \chi^k \). Since \( k \in N_{i_c}(\overline{R}) \), by Pareto optimality, \( N = N_{i_c}(\overline{R}) \).

Consider the \( i_1 \)-deviation \( \overline{R}^1 \) from \( R \) with \( R_{i_1}^1 = \overline{R}_{i_1} \). Then, by Lemma 3 and strategy-proofness, \( \varphi_{i_1}(R) = \varphi_{i_1}(R^1) = 0 \). Hence, by weak non-bossiness, \( \varphi(R) = \varphi(R^1) \). Next, consider the \( i_2 \)-deviation \( R^2 \) from \( R^1 \) with \( R_{i_2}^1 = \overline{R}_{i_2} \). Similarly as before, it follows that \( \varphi(R^1) = \varphi(R^2) \). Hence, \( \varphi(R^2) = \varphi(R) \). Applying this argument several times yields for the profile \( R^{i_{c-1}} \) such that \( R_{i_1}^{i_{c-1}} = \overline{R}_{i_1}^{i_{c-1}} \) and \( R_{i_2}^{i_{c-1}} = \overline{R}_{i_2}^{i_{c-1}} \), \( \varphi(R^{i_{c-1}}) = \varphi(\overline{R}^{i_{c-1}}) = 0 \). Note that \( \overline{R} \) is a \( j \)-deviation from \( R^{i_{c-1}} \). By strategy-proofness, \( \varphi_j(\overline{R}) = \Omega \). This is a contradiction to \( \varphi(\overline{R}) = \Omega \chi^k, k \neq j \).
Finally, we complete the proof of Theorem 1.

**Proof of Theorem 1.** By Lemma 2, (i) implies (ii). By Lemma 4, (ii) implies (i). It is easy, but tedious, to prove that a rule as described in (iii) satisfies Pareto optimality,coalitional strategy-proofness, and therefore strategy-proofness and weak non-bossiness. It remains to prove that either (i) implies (iii) or (ii) implies (iii). We prove that (i) implies (iii).

By Lemmas 3 and 4, it follows that \( \mathcal{R}_\varphi = \emptyset \). Hence, for all \( R \in \mathcal{D}^N \), \( \varphi(R) = \Omega \chi^i \) for some \( i \in N \).

(a) Definition of \( \pi \in \Pi^N \).

(a.1) Let \( R \in \mathcal{D}^N \) be such that \( N_\Omega(R) = N \). Then, \( \varphi(R) = \Omega \chi^i \) for some \( i \in N \). Set \( \pi(i) = 1 \). By coalitional strategy-proofness, for all \( \bar{R} \in \mathcal{D}^N \) with \( i \in N_\Omega(\bar{R}) \subseteq N \),

\[
\varphi(\bar{R}) = \Omega \chi^i.
\]  

(a.2) Next, let \( R \in \mathcal{D}^N \) be such that \( N_\Omega(R) = N^1 := N \setminus \{i_1\} \). Then, \( \varphi(R) = \Omega \chi^i \) for some \( i_2 \in N^1 \). Set \( \pi(i_2) = 2 \). By coalitional strategy-proofness, for all \( \bar{R} \in \mathcal{D}^N \) with \( i_2 \in N_\Omega(\bar{R}) \subseteq N^1 \),

\[
\varphi(\bar{R}) = \Omega \chi^i.
\]  

(a.3) Let \( R \in \mathcal{D}^N \) be such that \( N_\Omega(R) = N^2 := N^1 \setminus \{i_2\} \). Then \( \varphi(R) = \Omega \chi^i \) for some \( i_3 \in N^2 \). Set \( \pi(i_3) = 3 \), etc.

It is now clear how the linear order \( \pi \in \Pi^N \) is constructed step by step by leaving out the agent receiving the whole endowment. By the definition of \( \pi \), it follows that for all \( R \in \mathcal{D}^N \) such that \( N_\Omega(R) \neq \emptyset \),

\[
\varphi(R) = \Omega \chi^{\arg\min\{\pi(i) | i \in N_\Omega(R)\}}.
\]  

(b) Definition of \( g : \mathcal{D}^N \rightarrow N \).

Define a choice function \( g : \mathcal{D}^N \rightarrow N \) as follows. For all \( R \in \mathcal{D}^N \),

\[
g(R) := \begin{cases} 
    i & \text{if } \varphi(R) = \Omega \chi^i \text{ and } N_\Omega(R) = \emptyset, \\
    \min\{i | i \in N_{0,\Omega}(R)\} & \text{if } N_\Omega(R) \neq \emptyset \text{ and } N_{0,\Omega}(R) \neq \emptyset, \\
    \min\{i | i \in N_{0,\Omega}(R)\} & \text{if } N_\Omega(R) \neq \emptyset \text{ and } N_{0,\Omega}(R) = \emptyset.
\end{cases}
\]

Hence, by the definition of \( g \), it follows that for all \( R \in \mathcal{D}^N \) such that \( N_\Omega(R) = \emptyset \) and \( N_{0,\Omega}(R) \neq \emptyset \),

\[
\varphi(R) = \Omega \chi^{g(R)}.
\]  

Note, that the choice function \( g \) is not uniquely determined: (8) holds for all choice functions \( \bar{g} : \mathcal{D}^N \rightarrow N \) such that \( \bar{g}(R) = i \) if \( \varphi(R) = \Omega \chi^i \) and \( N_\Omega(R) = \emptyset \).
(c) Definition of $k \in N$.

Let $R \in D^N$ be such that $N_0(R) = N$. Then, $\varphi(R) = \chi^k_i$ for some $i \in N$. By coalitional strategy-proofness, it follows for all $\bar{R} \in D^N$ with $N_0(\bar{R}) = N$, $\varphi(\bar{R}) = \chi^k_i$.

Hence, by setting $k = i$, for all $R \in D^N$ with $N_0(R) = N$,

$$\varphi(R) = \chi^k_i.$$

(9)

Now, Eqs. (7), (8), and (9) imply that for all $R \in D^N$, $\varphi(R) = \chi^{f_{i,\bar{e}}(R)}$.

Remark 1. As shown in Klaus et al. (1997) the class of rules described in Theorem 1 equals the class of rules that satisfy Pareto optimality, strategy-proofness, and the solidarity property “replacement-domination.” The latter property requires that if an agent’s preference relation is “replaced” by some other admissible preference relation, then this unilateral change affects the remaining agents in the same direction, i.e., these agents all (weakly) gain or they all (weakly) lose.

In the next example, we describe a rule that satisfies all properties stated in Theorem 1.

Example 3. The following rule $\varphi^{\min}$ satisfies Pareto optimality, coalitional strategy-proofness, and weak non-bossiness. For all $R \in D^N$:

Case 1. If $N_0(R) \neq \emptyset$, then $\varphi^{\min}(R) = \chi^{\min\{i \mid i \in N_0(R)\}}$.

Case 2. If $N_0(R) = \emptyset$ and $N_{0,0}(R) \neq \emptyset$, then $\varphi^{\min}(R) = \chi^{\min\{i \mid i \in N_{0,0}(R)\}}$.

Case 3. If $N = N_0(R)$, then $\varphi^{\min}(R) = \chi^{\min\{i \mid i \in N_0(R)\}}$.

By $id \in \Pi$ we denote the identity permutation defined by $id(i) = i$ for all $i \in N$. Let $\pi = id$ and define $g^{\min} \in G^N$ such that for all $R \in D^N$,

$g^{\min}(R) = \min\{i \mid i \in N_{0,0}(R)\}$ if $N_{0,0}(R) \neq \emptyset$ and $g^{\min}(R) = \min\{i \mid i \in N\}$ otherwise. Furthermore, let $k^{\min} = \min\{i \mid i \in N\}$. Then, for all $R \in D^N$, $\varphi^{\min}(R) = \chi^{id \cdot g^{\min} \cdot k^{\min}(R)}$.

The following examples show that the characterization given in Theorem 1 is tight, i.e., dropping any of the properties yields alternative rules.

Example 4. The following rule $\check{\varphi}$ satisfies (coalitional) strategy-proofness and weak non-bossiness, but not Pareto optimality. For all $R \in D^N$, $\check{\varphi}(R) = \chi^{\min\{i \mid i \in N\}}$.

Example 5. The following rule $\hat{\varphi}$ satisfies Pareto optimality and weak non-bossiness, but not (coalitional) strategy-proofness. For all $R \in D^N$ such that $N_0(R) \neq \emptyset$, let $N^1_0(R) = \{i \in N_0(R) \mid d(R) = 0\}$ and $N^2_0(R) = \{i \in N_0(R) \mid d(R) \neq 0\}$. Then, for all $R \in D^N$, we define $\hat{\varphi}(R)$ as follows. If $N^1_0(R) \neq \emptyset$, then $\hat{\varphi}(R) = \chi^{\min\{i \mid i \in N^1_0(R)\}}$, if $N^1_0(R) = \emptyset$ and $N^2_0(R) \neq \emptyset$, then $\hat{\varphi}(R) = \chi^{\min\{i \mid i \in N^2_0(R)\}}$, and $\hat{\varphi}(R) = \varphi^{\min}(R)$ otherwise.
Without loss of generality, we define properties: anonymity, no-envy compatible with the fairness properties introduced above.

For all remaining \( R \in \mathcal{D}^N \), \( \tilde{\varphi}(R) = \varphi^{\text{min}}(R) \). \( \Diamond \)

**Incompatibilities**

Note that in all examples demonstrating incompatibilities Pareto optimality can easily be replaced by weak Pareto optimality.

We consider the compatibility of Pareto optimality with various fairness properties: anonymity, no-envy, and equal treatment of equals.\(^5\)

Anonymity requires that the agents' allotments do not depend on their names. No-envy states that no agent strictly prefers the allotment of another agent to his own allotment. Equal treatment of equals, a weakening of no-envy and of anonymity, requires that if two agents have the same preference relations, then each of them is indifferent between his allotment and the other agent's allotment.

Let \( \pi \in \Pi^N \) be a permutation on \( N \) and \( R \in \mathcal{D}^N \). Then, by \( R_\pi \) we mean \( (R_{\pi(i)})_{i \in N} \).

Anonymity. For all \( R \in \mathcal{D}^N \), all \( \pi \in \Pi^N \), and all \( i \in N \), \( \varphi_i(R_\pi) = \varphi_{\pi(i)}(R) \).

No-Env. For all \( R \in \mathcal{D}^N \), and all \( i, j \in N \), \( \varphi_i(R)R_i\varphi_j(R) \).

Equal Treatment of Equals. For all \( R \in \mathcal{D}^N \), and all \( i, j \in N \), if \( R_i = R_j \), then \( \varphi_i(R)I_i\varphi_j(R) \).

By the next example, we show that Pareto optimality is generally not compatible with the fairness properties introduced above.

**Example 7.** Let \( N = \{1, 2\} \), \( \Omega = 1 \), and \( R \in \mathcal{D}^N \) be such that \( R_1 = R_2 \) and \( d(R_1) = 0.7, 0I_10.9, \) and \( 0.2I_10.8 \). Suppose the rule \( \varphi \) satisfies Pareto optimality. Then, by Lemma 1, for \( i = 1, 2 \), either \( \varphi_i(R) = 0 \) or \( \varphi_i(R) \in (0.9, 1] \). Since \( \varphi_2(R) = 1 - \varphi_1(R) \), it follows, by feasibility, that \( \varphi(R) \in \{(1, 0), (0, 1)\} \). If \( \varphi \) is anonymous, then \( \varphi(R) = (0.5, 0.5) \). If \( \varphi \) satisfies no-envy or equal treatment of equals, then \( \varphi(R) \in \{(0.5, 0.5), (0.2, 0.8), (0.8, 0.2)\} \). However, none of these allocations is Pareto optimal. \( \Diamond \)

\(^5\)For an analysis of these properties for the problem of fair allocation when preferences are single-peaked, we refer to Ching (1992, 1994) and Sprumont (1991).
The next example shows that *Pareto optimality* and *continuity with respect to preferences* are not compatible. In the example we apply *continuity with respect to preferences* in an informal way: continuous changes of the preferences imply continuous changes in the allocations assigned by the rule. For the problem of fair allocation when preferences are single-peaked, *continuity with respect to preferences* was introduced by Sprumont (1991). This definition can easily be adjusted for rules when preferences are single-dipped.

**Example 8.** Let \( N = \{1, 2\} \), \( \Omega = 1 \), and \( R = (R_1, R_2) \in \mathcal{D}^N \) be such that \( 0I_11 \) and \( 1P_20 \). Suppose the rule \( \varphi \) satisfies *Pareto optimality*. Then, \( \varphi(R) = (0, 1) \). Now, change \( R_2 \) continuously into \( R_1 \) in such a way that in these changes agent 1 still prefers 1 to 0. *Continuity with respect to preferences* implies (in the limit) \( \varphi(R_1, R_1) = (0, 1) \). Next, consider \( \bar{R} = (R_2, R_1) \in \mathcal{D}^N \). Then, by the same argument, it follows that \( \varphi(\bar{R}) = (1, 0) \) and (in the limit) \( \varphi(R_1, R_1) = (1, 0) \). This is in contradiction to \( \varphi(R_1, R_1) = (0, 1) \).

4. THE ASSIGNMENT OF AN INDIVISIBLE OBJECT

Consider the well-known problem of allocating an indivisible object among a group of agents, e.g., a task or a real object. Obviously, this problem is closely related to the allocation problem with single-dipped preferences we introduced in Section 2 (see Lemma 3).

In order to keep this section self-contained, we briefly introduce the model. An indivisible object \( \Omega \) has to be allocated among a non-empty and finite set \( N \) of agents. Each agent \( i \in N \) is equipped with a preference relation \( R_i \) defined over the two alternatives “receiving nothing,” denoted by 0, and “receiving the object,” denoted by \( \Omega \). Hence, for each agent \( i \in N \) either \( 0P_i\Omega \), \( 0I_i\Omega \), or \( \Omega P_i0 \). By \( \mathcal{R}_{\{0, \Omega\}} \) we denote the set of preference relations over \( \{0, \Omega\} \) and \( \mathcal{R}_{\{0, \Omega\}}^N \) denotes the set of (preference) profiles \( R = (R_i)_{i \in N} \) such that for all \( i \in N \), \( R_i \in \mathcal{R}_{\{0, \Omega\}} \). Thus, the class of all economies is denoted by \( \mathcal{R}_{\{0, \Omega\}}^N \).

A feasible allocation for \( R \in \mathcal{R}_{\{0, \Omega\}}^N \) is an assignment of the object \( \Omega \) to an agent \( i \in N \); we denote this allocation by \( \Omega \chi_i \). Note that it is without loss of generality that free disposal of the commodity is not allowed. An *assignment rule* \( \varphi \) is a function that assigns to every \( R \in \mathcal{R}_{\{0, \Omega\}}^N \) a feasible allocation, denoted \( \varphi(R) \). Note that either \( \varphi_i(R) = 0 \) or \( \varphi_i(R) = \Omega \). Properties of assignment rules and further notation are as defined in Sections 2 and 3.

It is easy to show that Theorem 1 and Corollary 1 remain true for assignment rules. Before stating this result as a corollary, we show that *Pareto optimality* implies *strategy-proofness*. 
LEMMA 5. Let \( \varphi \) be an assignment rule that satisfies Pareto optimality. Then \( \varphi \) satisfies strategy-proofness.

Proof. Let \( \varphi \) satisfy Pareto optimality and suppose by contradiction that \( \varphi \) is not strategy-proof. Then there exist \( R, \bar{R} \in \mathcal{R}_{[0, \Omega]}^N \) and \( j \in N \) such that \( \bar{R} \) is a \( j \)-deviation from \( R \) and \( \varphi_j(\bar{R}) \neq \varphi_j(R) \). This can only occur if either
\[ (a) \quad j \in N_{\Omega}(R), \quad \varphi_j(R) = 0, \quad \text{and} \quad \varphi_j(\bar{R}) = \Omega, \] or
\[ (b) \quad j \in N_{\Omega}(\bar{R}), \quad \varphi_j(\bar{R}) = 0, \quad \text{and} \quad \varphi_j(R) = \Omega. \]

(a) Since \( \bar{R} \) is a \( j \)-deviation from \( R \) and \( \bar{R}_j \neq R_j \), either \( 0\Omega_j \Omega \) or \( 0\overline{\Omega}_j \Omega \). Since \( \varphi_j(R) = 0 \), there exists \( k \in N \) with \( \varphi_k(R) = \Omega \). Furthermore, by Pareto optimality, \( k \in N_{\Omega}(R) \) and \( k \in N_{\Omega}(\bar{R}) \). Hence, by Pareto optimality, \( \varphi_j(\bar{R}) = 0 \). This is a contradiction.

(b) Since \( j \in N_{\Omega}(R) \) and \( \varphi_j(R) = \Omega \), by Pareto optimality, \( N_{\Omega}(R) = N \).
Since \( \bar{R} \) is a \( j \)-deviation from \( R \) and \( \bar{R}_j \neq R_j \), either \( \Omega \bar{I}_j 0 \) or \( \Omega \bar{P}_j 0 \). Hence, by Pareto optimality, \( \varphi_j(\bar{R}) = \Omega \). This is a contradiction. \( \blacksquare \)

COROLLARY 2. Let \( \varphi \) be an assignment rule. Then the following statements are equivalent.

(i) \( \varphi \) satisfies Pareto optimality and coalitional strategy-proofness.

(ii) \( \varphi \) satisfies Pareto optimality and weak non-bossiness.

(iii) There exist \( \pi \in \Pi^N, g \in G^N \), and \( k \in N \) such that for all \( R \in \mathcal{R}_N \),
\[ \varphi(R) = \Omega [f^{g, k}(R)]. \]  \( (11) \)

The rule \( \varphi^{\min} \) as described in Example 3 is an example of an assignment rule that satisfies all properties stated in Corollary 2. Example 4 shows the independence of Pareto optimality from coalitional strategy-proofness and weak non-bossiness. However, in order to prove the independence of coalitional strategy-proofness and weak non-bossiness from Pareto optimality we need a new example.

EXAMPLE 9. The following rule \( \hat{\varphi} \) satisfies Pareto optimality but neither weak non-bossiness nor coalitional strategy-proofness. Without loss of generality, we define \( \hat{\varphi}(R) \) for \( N = \{1, 2, 3\} \). Let \( R \in \mathcal{R}_{\{0, \Omega\}}^N \). If \( N_{\Omega}(R) \neq \emptyset \) and \( |N_{\Omega}(R)| > 1 \), then
\[ \hat{\varphi}(R) = \begin{cases} \Omega \chi_{1}^{1} & \text{if } \Omega P_{1} 0, \\ \Omega \chi_{2}^{2} & \text{if } 0 I_{1} \Omega, \\ \Omega \chi_{3}^{3} & \text{if } 0 P_{1} \Omega, \end{cases} \]  \( (12) \)
and \( \hat{\varphi}(R) = \varphi^{\min}(R) \) otherwise. \( \Diamond \)
Remark 2. Pápai (1998) considers the problem of allocating an indivisible object where agents are not indifferent between the alternatives “receiving nothing” and “receiving the object” and where free disposal is allowed. For this model she proves that the class of rules that satisfy Pareto optimality, strategy-proofness, and non-bossiness (in terms of allotments) equals the class of serial dictatorships (Pápai, 1998, Proposition 2). Furthermore, Pápai (1998) studies the trade-off between the properties Pareto optimality, strategy-proofness, non-bossiness (in terms of allotments), and non-dictatorship.

Restricting our model to the model of Pápai (1998), the class of rules described in Corollary 2 equals the serial dictatorships described in Pápai (1998). The proofs of this result are different.

APPENDIX

Proof of Lemma 1. Let \( \varphi \) be a rule and let \( R \in \mathcal{D}^N \). It is easy to show that all allocations described in Lemma 1 are Pareto optimal. It remains to show that they are the only Pareto optimal allocations. Let \( \varphi \) be Pareto optimal and let \( y = \varphi(R) \).

Case 1. \( N_{\Omega}(R) \neq \emptyset \) and \( N_{0, \Omega}(R) \neq \emptyset \). It is easy to see that for all \( j \in N_{0, \Omega}(R) \), \( k \in N_{0}(R) \), \( \Omega \chi^{k} \) Pareto dominates \( \Omega \chi^{j} \). Now, suppose by contradiction that \( N = \{ i \in N \mid y_{i} \notin \{ 0, \Omega \} \} \neq \emptyset \). Let \( j \in N \). If \( j \in N_{0, \Omega}(R) \), then \( \Omega \chi^{j} \) is a Pareto improvement over \( y \): for all \( i \in N \), \( \Omega \chi^{j}_{i} R_{i} y_{i} \) and \( \Omega \chi^{j}_{i} P_{j} y_{i} \). If \( j \in N_{0}(R) \), then \( \Omega \chi^{k} \) where \( k \in N_{0, \Omega}(R) \) is a Pareto improvement over \( y \): for all \( i \in N \), \( \Omega \chi^{k}_{i} R_{i} y_{i} \) and \( \Omega \chi^{k}_{i} P_{j} y_{i} \).

Case 2. \( N_{\Omega}(R) = \emptyset \) and \( N_{0, \Omega}(R) \neq \emptyset \). It is easy to see that for all \( j \in N_{0, \Omega}(R) \), \( k \in N_{0}(R) \), \( \Omega \chi^{k} \) Pareto dominates \( \Omega \chi^{j} \). Now, suppose by contradiction that \( N = \{ i \in N \mid y_{i} \notin \{ 0, \Omega \} \} \neq \emptyset \). Let \( j \in N \). If \( j \in N_{0, \Omega}(R) \), then \( \Omega \chi^{j} \) is a Pareto improvement over \( y \): for all \( i \in N \), \( \Omega \chi^{j}_{i} R_{i} y_{i} \) and \( \Omega \chi^{j}_{i} P_{j} y_{i} \). If \( j \in N_{0}(R) \), then \( \Omega \chi^{k} \) where \( k \in N_{0, \Omega}(R) \) is a Pareto improvement over \( y \): for all \( i \in N \), \( \Omega \chi^{k}_{i} R_{i} y_{i} \) and \( \Omega \chi^{k}_{i} P_{j} y_{i} \).

Case 3. \( N_{\Omega}(R) = N \). Suppose by contradiction that \( N = \{ i \in N \mid y_{i} \notin \{ 0, \Omega \} \} \neq \emptyset \). Let \( j \in N \) and \( k \in N \) be such that \( y_{k} \neq 0 \). Then, \( \Omega \chi^{j} \) is a Pareto improvement over \( y \): for all \( i \in N \), \( \Omega \chi^{j}_{i} R_{i} y_{i} \) and \( \Omega \chi^{j}_{i} P_{j} y_{i} \).
REFERENCES