The strong sequential core for stationary cooperative games

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Abstract

Infinite time horizon stationary cooperative games are considered where at each date the instantaneous NTU-game is determined by the state of nature. The strong sequential core selects those utility streams that no coalition can improve upon by deviating at any moment in time. The main result of the paper states that the strong sequential core is non-empty provided that (i) the instantaneous NTU-games in all states are additively \( b \)-balanced, (ii) at least one of these games is strongly additively \( b \)-balanced, and (iii) the discount factor is close enough to one.

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1. Introduction

The issue of cooperation has a long history both in game theory and in economics. Traditionally, cooperation has been situated within static environments where time and uncertainty play no essential role. A few works, in contrast, have been concerned with a phenomenon known as dynamic cooperation: situations where cooperative decisions are to be made repeatedly during time. Due to the extreme diversity of the phenomenon in question no unifying theory of dynamic cooperation has emerged so far. Several attempts, however, have been made to develop a concept of the core that would adequately predict the outcome of dynamic cooperation for specific
economic or game-theoretic situations. In this paper we present a possible scenario of dynamic cooperation formalized to a stationary cooperative game and examine a particular solution concept for this game: the so-called strong sequential core.

Whenever cooperation takes place in a dynamic context, involved parties face plenty of possibilities to deviate from earlier agreements. A coalition of agents, for example, may consider to cooperate within the grand coalition at the initial stages of the game only to split off later on. In cases of uncertainty, where information becomes available over time, this is even more likely to occur. Therefore, the core-concept for a dynamic cooperative game has to take its time and information structure explicitly into account in order to preclude coalitional deviations at all stages and at all information sets that can potentially be achieved during the course of the game.

Sequential core concepts have been studied earlier in a number of works on dynamic cooperation. Gale (1978) has proposed a sequential core concept for a dynamic monetary economy. Becker and Chakrabarti (1995) and Koutsougeras (1998) have considered core concepts for a capital accumulation model and an asset market economy, respectively. Kranich et al. (2005) have distinguished strong and weak versions of the sequential core and studied both versions in a deterministic setting where the agents face a finite sequence of games with transferable utility. Subsequently, Predtetchinski et al. (2002, 2006) have applied the concepts of the strong and the weak sequential core to two-period economies where agents are given the possibility to exchange assets in the first period. In Predtetchinski et al. (2004) the strong sequential core has been studied in the framework of a stationary exchange economy.

In the paper of Filar and Petrosjan (2000) finite time horizon cooperative games are considered where at each stage the instantaneous TU-game depends deterministically on the instantaneous game at the previous stage as well as on the history of play. In Petrosjan (1993, 1977) a game of pursuit gives rise to a cooperative game with or without transferable utility that evolves in continuous time and can be influenced by the players through a number of controls. In these settings the authors address the question of dynamic stability (time consistency) of classical cooperative solution concepts such as the core and the NM-solution. A solution is said to be dynamically stable if it is robust to the objections at the beginning of the game as well as at any subsequent moment of time. Thus the idea of dynamic stability is essentially the same as that behind the concept of the strong sequential core.

In this paper we consider an infinite time horizon cooperative game. At each moment in time the game is in one of a finite number of states. The state determines the instantaneous NTU-game to be played at that moment. Transition from one state to the next occurs according to an exogenously given Markov process. A solution to a dynamic cooperative game must specify utility payoffs for all agents conditional on every finite history of the state of nature.

The classical core of a dynamic cooperative game consists of all those utility streams that no coalition can improve upon by deviating at the very beginning of the game for the rest of time. This core concept is characterized as the core of a static NTU-game comprising the expected present values of all feasible utility processes. The classical core is essentially a static concept, for it neglects the time and the information structure of the game and overlooks the possibility of coalitional deviations at later stages.

Assuming that any coalition of players is free to deviate at any moment of time, we consider the subgames of the original dynamic game, where a certain finite history of the state of nature is common knowledge among the players. The utility stream is said to be a strong sequential core-element if it belongs to the classical core of every such subgame. The idea of defining a subgame consistent solution as an intersection of the solutions of the subgames is not new: already in Petrosjan (1977, 1993) a similar principle is employed to define the concept of dynamic stability.
A version of the strong sequential core concept studied in this paper is an adaptation of the concept from Gale (1978), Kranich et al. (2005) and from Predtetchinski et al. (2002, 2004).

The central question of the paper is that of non-emptiness of the strong sequential core. Many robust examples of empty strong sequential core in the two-period economy are presented in Predtetchinski et al. (2002). A particularly sharp emptiness result is obtained for two-period finance economies with an incomplete set of available assets: in this class of economies the strong sequential core is generically empty. In contrast, for stationary exchange economies (Predtetchinski et al., 2004) the strong sequential core is non-empty given that the discount factor is sufficiently close to one. In this paper we show that the strong sequential core in a stationary cooperative game is non-empty, provided that

(i) the instantaneous NTU-games in all states satisfy an additive $b$-balancedness condition,
(ii) at least one of these NTU-games satisfies a strong additive $b$-balancedness condition, and
(iii) the discount factor is sufficiently close to one.

Both additive $b$-balancedness and strong additive $b$-balancedness strengthen Billera’s $b$-balancedness condition (Billera, 1970). A special form of additive $b$-balancedness condition was used earlier by Billera and Bixby (1973) and Billera (1974) to obtain a characterization of market games.

In more detail, the content of the paper is as follows. In Section 2 stationary cooperative games are presented and the assumptions of the model are discussed. In Section 3 a characterization of the strong sequential core is developed. We demonstrate that the classical core of the stationary cooperative game consists only of stationary utility streams, provided that all transition probabilities are positive and feasibility sets are strictly convex. This property of the core allows for a representation of every dynamic subgame by means of a finite-dimensional static evaluation game. We prove that the strong sequential core is non-empty if and only if the cores of the evaluation games have a non-empty intersection. The additive $b$-balancedness and strong additive $b$-balancedness conditions are discussed in Section 4. The non-emptiness result for the strong sequential core is presented in Section 5. More precisely, this result says that if a state-contingent utility allocation belongs to the core of a specific limit game and is not feasible for any of the proper coalitions, then it belongs to the strong sequential core whenever the discount factor is close to one. An example of empty strong sequential core is presented in Section 6.

2. Stationary cooperative games

Let $n$ be a positive integer. Denote by $N$ the set of integers $\{1, \ldots, n\}$. An $n$-person game with non-transferable utility, or an NTU-game for short, is a family of sets $V = \langle V(Q) \rangle_{Q \subseteq N}$ satisfying the following conditions:

(G1) If $Q \neq \emptyset$, then $V(Q)$ is a non-empty closed subset of $\mathbb{R}^n$; $V(\emptyset) = \emptyset$.
(G2) $[x \in V(Q), y \in \mathbb{R}^n, y^i \leq x^i \text{ for all } i \in Q]$ implies $[y \in V(Q)]$.
(G3) There exists a vector $m = (m^i)_{i \in N}$ in $\mathbb{R}^n$ such that $V(\{i\}) = \{x \in \mathbb{R}^n \mid x^i \leq m^i\}$ for all $i \in N$, and the set $V(N) \cap \{x \in \mathbb{R}^n \mid x \geq m\}$ is non-empty and compact.

A stationary cooperative game is given by the tuple

$$\Gamma = \langle N, S, (U_s)_{s \in S}, \pi, \delta \rangle.$$
The ingredients of a stationary cooperative game $\Gamma$ are as follows. The set $N$ is the set of players and $S = \{1, \ldots, S\}$ is the set of states. The symbol $U_s$ denotes the instantaneous non-transferable utility game $\langle U_s(Q) \rangle_{Q \subseteq N}$ in the state $s \in S$. The matrix $\pi = (\pi(\sigma | s))_{\sigma, s \in S}$ is a column stochastic matrix of transitional probabilities: for any $\sigma$ and $s$ in $S$, $\pi(\sigma | s)$ is the transition probability from state $s$ to state $\sigma$. We assume that the players’ preferences over utility streams can be represented by the expected discounted utility function. The discount factor is given by $\delta \in (0, 1)$.

Let $C$ be a non-empty convex subset of $\mathbb{R}^n$. Then the recession cone $\mathcal{O}^+ [C]$ of $C$ is a set consisting of all vectors $y \in \mathbb{R}^n$ such that $C + y \subseteq C$. The lineality space of $C$ is a set of all vectors $y \in \mathbb{R}^n$ such that $C + y = C$. The matrix $\pi$ is said to be irreducible if it is not possible to partition the set $S$ in two subsets $S_1$ and $S_2$ in such a way that $\pi(\sigma | s) = 0$ for all $s \in S_1$ and all $\sigma \in S_2$.

We make the following assumption:

**Assumption (A).**

(A1) $U_s$ is an NTU-game for all $s \in S$.

(A2) The set $U_s(Q)$ is convex for all $s \in S$ and $Q \subseteq N$.

(A3) For each $Q \subseteq N$, if $z_1, \ldots, z_S$ are vectors such that $z_s \in \mathcal{O}^+[U_s(Q)]$ and $z_1 + \cdots + z_S = 0$, then $z_s$ belongs to the lineality space of $U_s(Q)$ for all $s \in S$.

(A4) $(\mathcal{O}^+[U_1(N)] + \cdots + \mathcal{O}^+[U_S(N)]) \cap \mathbb{R}^n_+ = \{0\}$.

(A5) The matrix $\pi$ is irreducible.

Assumption (A1) requires the collection of sets $U_s$ to satisfy conditions (G1)–(G3).

The convexity assumption (A2) is essential for most of our results. It ensures that a coalition that is able to improve upon a given utility stream can do so using a stationary utility stream. This property of the model makes it possible to restrict attention only to stationary utility streams and characterize a subgame of the stationary cooperative game by a static evaluation game. Without the convexity assumption, non-emptiness of the classical core (let alone the non-emptiness of the strong sequential core) becomes a considerably more complicated problem.

Assumptions (A3) and (A4) guarantee that the sum of the NTU-games $U_s$ over $s \in S$ is an NTU-game. They ensure that the sum of the sets $U_s(Q)$ is a closed set for each coalition $Q$, and that the sum of the sets $U_s(N)$ intersected with all individually rational payoffs is bounded.

In particular, assumptions (A3)–(A4) are satisfied if there exists a real number $M$ such that for all $Q \subseteq N$, $s \in S$, $u \in U_s(Q)$, and $i \in Q$ the inequality $u^i \leq M$ holds. In this case the recession cone of the set $U_s(Q)$ in given by

$$\mathcal{O}^+[U_s(Q)] = \{ z \in \mathbb{R}^n \mid z^i \leq 0 \text{ for all } i \in Q \},$$

and the lineality space of $U_s(Q)$ is the set

$$\{ z \in \mathbb{R}^n \mid z^i = 0 \text{ for all } i \in Q \}.$$

Of course, assumptions (A3)–(A4) are much more general. However, they restrict the ways in which the sets $U_s(Q)$ may be unbounded.

Assumption (A5) ensures that for all $s$ and $\sigma$ in $S$ there is a non-zero probability of ever reaching state $\sigma$ from state $s$.

Assumption (A) is sufficient for most of our results. For Theorem 2 and Corollary 1, however, the following strengthening of (A) is needed:

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1 For a comprehensive study of recession cones and related concepts see Rockafellar (1997).
Assumption (A*). Assumptions (A1)–(A5) are satisfied and, in addition,
(A6) The set \( U_s(N) \) is strictly convex for all \( s \in S \).
(A7) \( \pi(\sigma|s) > 0 \) for all \( s, \sigma \in S \).

A stationary cooperative game is a game with infinite time horizon, so the set of time periods or dates is \( T = \{0, 1, \ldots, t, \ldots\} \). At each date \( t \in T \) the state takes on one of the values from the set \( S \) determining the instantaneous NTU-game to be played at date \( t \). The state follows a Markov process with one-step transitional probabilities given by \( \pi \). The state at date \( t = 0 \) is known to be \( s_0 \in S \).

The set
\[
\mathbb{D} = \{(t, s_0, s_1, \ldots, s_t) \mid t \in T, s_t \in S \text{ for all } t \in \{1, \ldots, t\}\}
\]
consists of all possible finite histories of the state of nature. A typical element \( \xi = (t, s_0, s_1, \ldots, s_t) \) of the set \( \mathbb{D} \) summarizes the information available to the players at time \( t \). This information includes the current date \( t \) as well as the history \( (s_0, s_1, \ldots, s_t) \) of the state of nature up to the moment \( t \). The state of nature \( s_t \) at the current date \( t \) is referred to as a **current state**. The mapping \( t: \mathbb{D} \to T \) assigns to each finite history \( \xi \in \mathbb{D} \) the current date \( t(\xi) \). The mapping \( s: \mathbb{D} \to S \) assigns to each finite history \( \xi \) the current state \( s(\xi) \).

A given finite history \( \xi = (t, s_0, s_1, \ldots, s_t) \) gives rise to a set of histories that agree with \( \xi \) up to date \( t \). Formally, it is defined as
\[
\mathbb{D}(\xi) = \{(\tau, \sigma_0, \sigma_1, \ldots, \sigma_\tau) \in \mathbb{D} \mid \tau \geq t, \sigma_k = s_k \text{ for all } k \in \{0, 1, \ldots, t\}\}
\]
The probability that the state of nature will develop according to \( \eta = (\tau, \sigma_0, \sigma_1, \ldots, \sigma_\tau) \) conditional on the history \( \xi = (t, s_0, s_1, \ldots, s_t) \) up to date \( t \) can be computed using the formula:
\[
\rho(\eta|\xi) = \begin{cases} 1, & \text{if } \eta = \xi, \\ \times_{k=t+1}^\tau \pi(\sigma_k|\sigma_{k-1}), & \text{if } \eta \in \mathbb{D}(\xi) \text{ and } \tau > t, \\ 0, & \text{otherwise}. \end{cases}
\]

A function \( \mathcal{U} \) from \( \mathbb{D} \) into \( \mathbb{R}^n \) will be referred to as a **utility stream**. We write \( \mathcal{U}_\xi \) to denote the values for the function \( \mathcal{U} \). The expected present value of the utility stream \( \mathcal{U} \) conditional on the history \( \xi \in \mathbb{D} \) is given by the function
\[
\mathcal{V}_\xi(\mathcal{U}) = (1 - \delta) \sum_{\eta \in \mathbb{D}(\xi)} \rho(\eta|\xi) \delta^{t(\eta) - t(\xi)} \mathcal{U}_\eta.
\]
Given a stationary cooperative game \( \Gamma \) and a finite history \( \xi \) of the state of nature, let \( \Gamma_{\mathbb{D}(\xi)} \) denote the subgame of \( \Gamma \) that follows the realizations of the state of nature as specified in \( \xi \).

We continue with the definitions of the classical core for the subeconomy \( \Gamma_{\mathbb{D}(\xi)} \) and of the strong sequential core for the economy \( \Gamma \). Essentially, the former definition requires that no coalition be able to make all its members better off by deviating conditional on the history \( \xi \). The latter definition requires that for no particular history of the state of nature can a coalition of players improve upon a proposed allocation. In this way, the strong sequential core eliminates any incentives for coalitions to first agree to a proposed allocation but break the agreement later on.

**Definition 1.** The utility stream \( \mathcal{U} \) belongs to the **classical core** \( \mathcal{C}(\Gamma_{\mathbb{D}(\xi)}) \) of the sub-game \( \Gamma_{\mathbb{D}(\xi)} \) if

1. \( \mathcal{U}_\eta \in U_s(\eta)(N) \) for all \( \eta \in \mathbb{D}(\xi) \), and \( \mathcal{V}_\xi(\mathcal{U}) \) is finite;
(2) there exists no coalition \( Q \subseteq N \) and no utility stream \( Y \), such that \( Y_\eta \in U_{s(\eta)}(Q) \) for all \( \eta \in D(\xi) \), \( \forall_\xi (Y) \) is finite, and \( V_\xi^i(Y) > V_\xi^i(U) \) for all \( i \in Q \).

**Definition 2.** The **strong sequential core** of the stationary cooperative game \( \Gamma \) is the set of utility streams

\[
SSC(\Gamma) := \bigcap_{\xi \in D} C(\Gamma_{D(\xi)}).
\]

3. A characterization of the strong sequential core

The characterizations derived in this section make use of stationary utility streams and a closely related concept of an evaluation game. A utility stream \( U \) is said to be **stationary** if its value \( U_\xi \) at \( \xi \in D \) depends only on the current state \( s(\xi) \) and does not depend on time or the history of the state of nature preceding the current date \( t(\xi) \). For any stationary utility stream \( U \), there is an \((nS)\)-dimensional vector \( u = (u_1, \ldots, u_S) \) with \( u_s \in \mathbb{R}^n \) such that \( U_\xi = u_s(\xi) \) for all \( \xi \in D \); we write \( U = \{u\} \).

Let \( IS \) denote the \( S \)-dimensional identity matrix. Observe that the matrix \( IS - \delta \pi \) is invertible, since the spectral radius of the matrix \( \pi \) equals \( \delta < 1 \). Let \( \psi = (1 - \delta)(IS - \delta \pi)^{-1} \), and for every pair of states \( \sigma \) and \( s \) let \( \psi(\sigma | s) \) denote an element of the matrix \( \psi \) at the intersection of row \( \sigma \) and column \( s \). Observe that the elements in a given column of the matrix \( \psi \) add up to 1. As we assume \((A5)\) matrix \( \pi \) to be irreducible, \( \psi(\sigma | s) > 0 \) for all \( \sigma, s \in S \).

**Definition 3.** The **evaluation game** \( \Omega_s \) is a cooperative game where the possibilities of a coalition \( Q \) are represented by the set \( U_1(Q) \times \cdots \times U_S(Q) \), and the players’ expected utilities from consuming the state-contingent utility tuple \( u = (u_1, \ldots, u_S) \) with \( u_s \in \mathbb{R}^n \) such that \( U_\xi = u_s(\xi) \) for all \( \xi \in D \); we write \( U = \{u\} \).

Let \( IS \) denote the \( S \)-dimensional identity matrix. Observe that the matrix \( IS - \delta \pi \) is invertible, since the spectral radius of the matrix \( \delta \pi \) equals \( \delta < 1 \). Let \( \psi = (1 - \delta)(IS - \delta \pi)^{-1} \), and for every pair of states \( \sigma \) and \( s \) let \( \psi(\sigma | s) \) denote an element of the matrix \( \psi \) at the intersection of row \( \sigma \) and column \( s \).

**Proposition 1.** Let \( \xi \in D \) be a finite history of the state of nature with the current state \( s = s(\xi) \) and let \( U = \{u\} \) be a stationary utility stream. Then \( V_\xi(U) = v_s(u) \).

Proposition 1 is a well-known result. It makes it possible to interpret the evaluation game \( \Omega_s(\xi) \) as a static counterpart of the dynamic subgame of \( \Gamma \) starting at \( \xi \) where each coalition is confined to use stationary utility streams only. Indeed, the set \( U_1(Q) \times \cdots \times U_S(Q) \) essentially consists of all stationary utility streams feasible for the coalition \( Q \), while the utility function \( v_s(\xi) \) gives the expected present value of a stationary utility stream \( \{u\} \) conditional on the history \( \xi \).

Associated with each evaluation game is the NTU-game \( V_s \) that is essentially an image of \( \Omega_s \) under the function \( v_s \). More precisely, define \( V_s(Q) \) as the union of the vectors \( v_s(u) \) as \( u \) ranges in the set \( U_1(Q) \times \cdots \times U_S(Q) \). Thus, each point of \( V_s(Q) \) is the expected present value of some \( Q \)-feasible stationary utility stream. We can write \( V_s(Q) \) as a sum

\[
V_s(Q) = \sum_{\sigma \in S} \psi(\sigma | s)U_\sigma(Q).
\]
Clearly, the state-contingent utility tuple \( u \) is in the core of \( G_s \) if and only if \( v_s(u) \) is in the core of \( V_s \).

Proposition 2 below states that the collection \( V_s \) satisfies conditions \((G1)–(G3)\). In particular, the assumptions \((A3)\) and \((A4)\) guarantee that \( V_s(Q) \) is a closed set, and that the intersection of \( V_s(N) \) with the set of individually rational payoffs is bounded.

**Proposition 2.** Let \( \Gamma \) be a stationary cooperative game satisfying Assumption \((A)\). Then \( V_s \) is an NTU-game.

**Proof.** Let \( Q \) be a non-empty subset of \( N \). Then \( V_s(Q) \) is obviously a non-empty set. To prove that \( V_s(Q) \) is a closed set, observe that for any positive number \( \lambda \) and any convex subset \( C \) of \( \mathbb{R}^n \), the recession cone of the set \( \lambda C \) coincides with that of the set \( C \). In particular,

\[
\mathcal{O}^+\left[ \psi(\sigma |s)U_\sigma (Q) \right] = \mathcal{O}^+\left[ U_\sigma (Q) \right].
\]

From this, from Assumption \((A3)\) and Corollary 9.1.1 in Rockafellar (1997, pp. 74–75), \( V_s(Q) \) is a closed set and its recession cone is given by

\[
\mathcal{O}^+\left[ V_s(Q) \right] = \sum_{\sigma \in S} \mathcal{O}^+\left[ U_\sigma (Q) \right].
\]

Condition \((G2)\) and the first part of condition \((G3)\) can be easily verified. To prove that the set \( V_s(N) \cap \{ v \in \mathbb{R}^n \mid v \geq m \} \) is bounded we show its recession cone to contain only a zero vector. By Theorem 8.4 from Rockafellar (1997, p. 64), any non-empty closed convex set whose recession cone consists only of the zero vector is bounded. By Corollary 8.3.3 from Rockafellar (1997, p. 64), if the intersection of two closed convex sets is non-empty, then its recession cone equals the intersection of the respective recession cones. Thus, we compute:

\[
\mathcal{O}^+\left[ V_s(N) \right] \cap \mathbb{R}^n_+ = \sum_{\sigma \in S} \mathcal{O}^+\left[ U_\sigma (N) \right] \cap \mathbb{R}^n_+ = \{0\}.
\]

The last of these equalities follows from assumption \((A4)\). \( \square \)

Theorem 1 provides a criterion for a given stationary utility stream to be an element of the strong sequential core of a stationary cooperative game.

**Theorem 1.** Let \( \Gamma \) be a stationary cooperative game satisfying Assumption \((A)\). Let a stationary utility stream \( U = \{u\} \) be given. Then

(1) \( U \in C(\Gamma_{D(\xi)}) \) if and only if \( u \in C(\mathcal{G}_s(\xi)) \).

(2) \( U \in SSC(\Gamma) \) if and only if \( u \in C(\mathcal{G}_s) \) for all \( s \in S \).

**Proof.** To prove the first part of Theorem 1 we demonstrate that a coalition \( Q \subseteq N \) is able to improve upon \( U \) in the subgame \( \Gamma_{D(\xi)} \) if and only if it can improve upon \( u \) in the evaluation game \( G_s(\xi) \). The second part of Theorem 1 is an immediate corollary of the first part.

Suppose that there is a deviation \( y \) from \( u \) by a coalition \( Q \) in the evaluation game \( G_s(\xi) \), that is \( y \in \times_{\sigma \in S} U_\sigma (Q) \) and \( v_{s(\xi)}^i(y) > v_{s(\xi)}^i(u) \) for all \( i \in Q \). Then the stationary utility stream \( \{y\} \) is a deviation from the utility stream \( \{u\} \) in the subgame \( \Gamma_{D(\xi)} \).
Suppose that there is a deviation from \( \{u\} \) by a coalition \( Q \) in the subgame \( \Gamma_{\mathcal{D}(\xi)} \), i.e. the utility stream \( \mathcal{Y} \) such that \( \mathcal{Y}_\eta \in U_s(\eta)(Q) \) for all \( \eta \in \mathcal{D}(\xi) \), \( \mathcal{V}_\xi(\mathcal{Y}) \) is finite, and \( \mathcal{V}_\xi(\mathcal{Y}) > \mathcal{V}_\xi(\{u\}) \) for all \( i \in Q \). Note that the utility stream \( \mathcal{Y} \) need not be stationary. However, as we demonstrate in the remainder of the proof, coalition \( Q \) can improve upon \( \mathcal{U} \) using a stationary utility stream \( \{z\} \). Convexity of the feasibility sets plays the central role in our argument.

Choose the vectors \( y_s \) arbitrarily in \( U_s(Q) \) for all \( s \in S \). For every date \( T \in T \) define the process \( \mathcal{Y}(T) \) by

\[
\mathcal{Y}_\eta(T) = \begin{cases} \mathcal{Y}_\eta, & \text{if } t(\eta) < T, \\ y_s(\eta), & \text{otherwise.} \end{cases}
\]

Observe that the process \( \mathcal{Y}(T) \) is stationary as of date \( T \) and therefore is bounded. Since the expected present value of \( \mathcal{Y}(T) \) converges to that of \( \mathcal{Y} \) as \( T \) goes to infinity, there is a date \( T_0 \in T \) such that

\[
\mathcal{V}_\xi(\mathcal{Y}(T_0)) > \mathcal{V}_\xi(\{u\}) \quad \text{for all } i \in Q.
\]

Next, define the utility tuples \( z_\sigma \) for all \( \sigma \in S \) by the equation

\[
z_\sigma = \frac{1 - \delta}{\psi(\sigma|s(\xi))} \sum_{\eta \in \mathcal{D}(\xi)} \rho(\eta|\xi) \delta(t(\eta) - t(\xi)) \mathcal{Y}_\eta(T_0).
\]

Because the process \( \mathcal{Y}(T_0) \) is bounded, the sum in the above equation is well-defined. Therefore, \( z_\sigma \) is a weighted average of the values for the process \( \mathcal{Y}(T_0) \) over all histories \( \eta \) in \( \mathcal{D}(\xi) \) with \( s(\eta) = \sigma \). For all such histories, \( \mathcal{Y}_\eta(T_0) \in U_\sigma(Q) \). Since the set \( U_\sigma(Q) \) is closed and convex, \( z_\sigma \in U_\sigma(Q) \). We compute:

\[
\mathcal{V}_\xi(\{z\}) = \sum_{\sigma \in S} \psi(\sigma|s(\xi)) z_\sigma
\]

\[
= (1 - \delta) \sum_{\sigma \in S} \sum_{\eta \in \mathcal{D}(\xi)} \rho(\eta|\xi) \delta(t(\eta) - t(\xi)) \mathcal{Y}_\eta(T_0)
\]

\[
= (1 - \delta) \sum_{\eta \in \mathcal{D}(\xi)} \rho(\eta|\xi) \delta(t(\eta) - t(\xi)) \mathcal{Y}_\eta(T_0)
\]

\[
= \mathcal{V}_\xi(\mathcal{Y}(T_0)).
\]

We see that the stationary utility stream \( \{z\} \) is a deviation from \( \{u\} \) by the coalition \( Q \) in the subgame \( \Gamma_{\mathcal{D}(\xi)} \). Therefore, \( z \) is a deviation from \( u \) by \( Q \) in the evaluation game \( g_{s(\xi)} \).

Theorem 2 gives sufficient conditions for the classical core of the stationary cooperative game to contain only stationary utility streams.

**Theorem 2.** Let \( \Gamma \) be a stationary cooperative game satisfying Assumption (\( A^* \)). Then all utility streams in the classical core of \( \Gamma \) are stationary.

**Proof.** Consider an \( N \)-feasible utility stream \( \mathcal{U} \) with a finite expected present value at time \( t = 0 \).

If \( \mathcal{U} \) is not stationary, then there exist two histories \( \lambda, \zeta \in \mathcal{D} \) such that \( s(\lambda) = s(\zeta) \) and \( \mathcal{U}_\lambda \neq \mathcal{U}_\zeta \).

Note that both \( \mathcal{U}_\lambda \) and \( \mathcal{U}_\zeta \) are the elements of the set \( U_{s(\lambda)}(N) \). Under Assumption (\( A8 \), the
probabilities $\rho(\lambda)$ and $\rho(\varsigma)$ of $\lambda$ and $\varsigma$ conditional on the initial history $(0, s_0)$ are both positive. Strict convexity of the set $U_{s(\lambda)}(N)$ therefore implies that the vector

$$u = \frac{\delta^t(\lambda) \rho(\lambda) U_\lambda + \delta^t(\varsigma) \rho(\varsigma) U_\varsigma}{\delta^t(\lambda) \rho(\lambda) + \delta^t(\varsigma) \rho(\varsigma)}$$

belongs to the interior of $U_{s(\lambda)}(N)$. Therefore, there exists a vector $y \in U_{s(\lambda)}(N)$ such that $y^i > u^i$ for all $i \in N$. Define the utility stream $Y$ as follows:

$$Y_\eta = \begin{cases} U_\eta, & \text{if } \eta \neq \lambda, \varsigma; \\ y, & \text{if } \eta = \lambda, \text{ or } \eta = \varsigma. \end{cases}$$

The utility stream $Y$ is feasible for the coalition $N$, the expected present value of $Y$ is finite, and is greater than that of the utility stream $U$ at $t = 0$, for each player in $N$. Hence, $Y$ is an improvement upon $U$ by the grand coalition at date $t = 0$. \Box

We conclude this section with a characterization of the strong sequential core.

**Corollary 1.** Let $\Gamma$ be a stationary cooperative game satisfying Assumption (A$^*$). Then

$$SSC(\Gamma) = \left\{ u \mid u \in \bigcap_{s \in S} C(\mathcal{G}_s) \right\}.$$

### 4. Additive $b$-balancedness and strong additive $b$-balancedness

In this section we consider two balancedness conditions: the so-called additive $b$-balancedness and strong additive $b$-balancedness. When applied to the instantaneous NTU-game $U_s$ for all $s \in S$, additive $b$-balancedness is sufficient for non-emptiness of the classical cores of all subgames of $\Gamma$. Moreover, if at least one of the NTU-games $U_s$ is strongly additively $b$-balanced and the discount factor $\delta$ is close enough to one, then also the strong sequential core is non-empty.

In what follows $\mathcal{N}^*$ denotes the collection of non-empty proper subsets of the set $N$. Let $b$ be a collection $\{b_Q^i \mid Q \in \mathcal{N}^*, \ i \in Q\}$ of non-negative real numbers such that $\sum_{i \in Q} b_Q^i > 0$ for each $Q \in \mathcal{N}^*$. Let $B$ denote the set of all such collections. A collection $\beta \subseteq \mathcal{N}^*$ of coalitions is said to be $b$-balanced, if there exist non-negative numbers $\lambda_Q$ for all $Q \in \beta$ such that

$$\sum_{Q \in \beta \cap \{i\}} \lambda_Q b_Q^i = 1, \quad \forall \ i \in N.$$

The numbers $\lambda_Q$ are called the balancing weights of the collection $\beta$. A particularly useful specification for the collection $b \in B$ is the following one:

$$b_Q^i = 1 \quad \text{for all } Q \in \mathcal{N}^* \text{ and } i \in Q. \quad (1)$$

**Definition 4.** Let $V$ be a game with non-transferable utility and $b$ be an element of the set $B$. The game $V$ is said to be additively $b$-balanced provided that the following condition is satisfied: If $\beta \subseteq \mathcal{N}^*$ is a $b$-balanced collection of coalitions, $\{\lambda_Q\}_{Q \in \beta}$ are balancing weights, $v_Q \in V(Q)$ for all $Q \in \beta$, then the vector $v_N \in \mathbb{R}^n$ defined by the equations

$$v_N^i = \sum_{Q \in \beta \cap \{i\}} \lambda_Q b_Q^i v_Q^i, \quad \forall \ i \in N \quad (2)$$

We conclude this section with a characterization of the strong sequential core.
belongs to the set $V(N)$. The NTU-game $V$ is said to be strongly additively $b$-balanced provided that any such vector $v_N$ belongs to the interior of $V(N)$.

A special case of the additive balancedness condition with $b$ specified as in Eq. (1) appears already in Billera and Bixby (1973) and in Billera (1974). As is discussed below, two well-known classes of games satisfy the additive balancedness condition: market games obtained from an underlying exchange economy (Example 1) and non-transferable utility games derived from an underlying normal form game using the concept of $\alpha$-efficiency as in Aumann (1961) (Example 2).

Example 1. Consider an exchange economy where the consumption set $X_i$ of player $i$ is a compact convex subset of an Euclidean space containing the initial endowment $e_i$ and the utility function is a continuous map $u^i : X^i \to \mathbb{R}$. In the associated market game $V$ the utility profile $v$ is feasible for a coalition $Q$ if there exist $x^i \in X^i$ for $i \in Q$ such that $\sum_{i \in Q} x^i = \sum_{i \in Q} e^i$ and $v^i \leq u^i(x^i)$ for all $i \in Q$. Under the maintained assumptions the market game $V$ is a well-defined non-transferable utility game. It is well known that the market game is balanced in the sense of Scarf (1967) provided that the utility functions are quasi-concave. As is demonstrated in Billera and Bixby (1973, Theorem 2.1), under a stronger assumption of concavity of the utility functions, the market game is additively balanced with respect to the collection $b$ given by Eq. (1). Notice, however, that a market game need not be strongly additively balanced. As follows from Proposition 4 below, if the core of the underlying exchange economy consist of a single point, then the associated market game is not strongly additively balanced for any $b \in B$.

To construct an NTU-game that is additively $b$-balanced for a given collection $b \in B$, we modify the definition of a market game as follows. Think of the exchange economy as being composed of $n$ types of players, with each type being represented by a continuum of identical individuals, and of the collection $b$ as fixing the rates of participation of players in each proper coalition. Thus for a proper coalition $Q \in N^*$ the feasibility condition assumes the form $\sum_{i \in Q} b^i_Q x^i = \sum_{i \in Q} b^i_Q e^i$, while for the grand coalition the feasible set is as in the market game. An argument similar to that in the proof of Theorem 2.1 in Billera and Bixby (1973) shows that the resulting NTU-game is additively $b$-balanced.

Example 2. Consider a normal form game where the strategy set $X^i$ of player $i$ is a compact convex subset of an Euclidean space and the utility function is a continuous map $u^i : \times_{i \in N} X^i \to \mathbb{R}$. Following Aumann (1961), we define a non-transferable utility game $V$ using the concept of $\alpha$-efficiency, as follows: A utility profile $v$ is feasible for a coalition $Q$ if there exists a joint strategy $x \in \times_{i \in Q} X^i$ for the members of coalition $Q$ such that $v^i \leq u^i(x, y)$ for all joint strategies $y \in \times_{j \in N \setminus Q} X^j$ of the outside players and for all $i \in Q$. Scarf (1971) shows the game $V$ to be Scarf-balanced provided that the utility functions are quasi-concave. Under a stronger assumption of concavity of the utility functions the game $V$ can be seen to be additively-balanced with respect to the collection $b$ given by Eq. (1).

The property of (strong) additive $b$-balancedness is additive over games in the following sense.

Proposition 3. Let $\Gamma$ be a stationary cooperative game satisfying Assumption (A), and let $b \in B$. Suppose that the instantaneous NTU-games $U_s$ are additively $b$-balanced in all states $s \in S$. Then the NTU-games $V_s$ are additively $b$-balanced for all $s \in S$. If, in addition, at least one
of the instantaneous NTU-games is strongly additively $b$-balanced, then the NTU-games $V_s$ are strongly additively $b$-balanced for all $s \in S$.

**Proof.** Let the $b$-balanced collection of coalitions $\beta \subseteq N^*$, the balancing weights $\{\lambda_Q\}_{Q \in \beta}$, and the vectors $v_Q \in V_s(Q)$ for all $Q \in \beta$ be given. Let the vector $v_N \in \mathbb{R}^n$ be defined by Eq. (2).

Choose the vectors $u_{Q\sigma} \in U_{\sigma}(Q)$ so that

$$v_Q = \sum_{\sigma \in S} \psi(\sigma \mid s)u_{Q\sigma}$$

for all $Q \in \beta$, and let the vectors $u_{N\sigma}$ be defined by the equations

$$u_{N\sigma} = \sum_{Q \in \beta \ni i} \lambda_Q b_Q^i u_{Q\sigma}^i, \quad \forall i \in N.$$

Then

$$v_N = \sum_{\sigma \in S} \psi(\sigma \mid s)u_{N\sigma}.$$

If the instantaneous games $U_{\sigma}$ are additively $b$-balanced in all states, then the vector $u_{N\sigma}$ belongs to the set $U_{\sigma}(N)$ for $\sigma \in S$, implying that $v_N$ is an element of the set $V_s(N)$. If, in addition, at least one of the instantaneous NTU-games, say the game $U_{\tilde{\sigma}}$, is strongly additively $b$-balanced, then $u_{N\tilde{\sigma}}$ is in the interior of the set $U_{\tilde{\sigma}}(N)$. Since the number $\psi(\tilde{\sigma} \mid s)$ is positive, this implies that the vector $v_N$ belongs to the interior of the set $V_s(N)$. 

It is straightforward to show that any additively $b$-balanced NTU-game $V$ is balanced in the sense of Billera (1970). That is, for any $b$-balanced collection of coalitions $\beta \subseteq N^*$ the inclusion

$$\bigcap_{Q \in \beta} V(Q) \subseteq V(N)$$

holds. It follows that an additively $b$-balanced game has a non-empty core. Let $C^*(V)$ denote the set of payoff vectors in the core of the NTU-game $V$ that are not feasible for proper coalitions. Formally,

$$C^*(V) = \partial V(N) \setminus \bigcup_{Q \in N^*} V(Q).$$

**Proposition 4.** If the NTU-game $V$ is strongly additively $b$-balanced, then the set $C^*(V)$ is non-empty.

**Proof.** Let $V$ be a strongly additively $b$-balanced NTU-game. For any $b$-balanced collection of coalitions $\beta \subseteq N^*$ let the set $W(\beta)$ be defined by

$$W(\beta) = \bigcap_{Q \in \beta} V(Q).$$

Define the NTU-game $\tilde{V}$ as follows. Let $\tilde{V}(Q) = V(Q)$ for all proper subsets $Q$ of $N$, and let $\tilde{V}(N)$ be the union of $W(\beta)$ over all $b$-balanced collections $\beta \subseteq N^*$. We leave the verification of conditions (G1)–(G3) to the reader. By construction, the NTU-game $\tilde{V}$ is $b$-balanced, and therefore has a non-empty core. Let $\tilde{v}$ be a core element of the game $\tilde{V}$. 


Since $V$ is strongly additively $b$-balanced, the inclusion $W(\beta) \subset \text{int} V(N)$ holds for any $b$-balanced collection $\beta$. It follows that $\bar{V}(N) \subset \text{int} V(N)$.

Consider the set

$$T = \{ t \in \mathbb{R} \mid (\bar{v} + t1_n) \in V(N) \},$$

where $1_n$ is a vector in $\mathbb{R}^n$ with all components equal to one. As $\bar{v}$ is an element of $\bar{V}(N)$, it is also an element of $V(N)$. Hence, $0 \in T$. Since $V(N)$ is closed, $T$ is also closed. Moreover, $T$ is bounded from above for otherwise the set $V(N)$ would coincide with $\mathbb{R}^n$, violating condition (G3). It follows that the supremum $\bar{t}$ of the set $T$ is finite, and is an element of $T$.

We show that the vector $v^* = \bar{v} + \bar{t}1_n$ belongs to the set $C^*(V)$. Since $\bar{t} \in T$, we have the inclusion $v^* \in V(N)$. If $v^*$ were an interior point of $V(N)$, then there would exist a $t > 0$ such that $(v^* + t1_n)$ is in $V(N)$. Hence, $(\bar{t} + t)$ would be an element of $T$, a contradiction.

Since $0 \in T$, we know that $\bar{t} \geq 0$. If $\bar{t}$ were equal to zero, then $v^* = \bar{v}$ would be an element of $\bar{V}(N)$ and therefore an interior point of $V(N)$. However, we already know that $v^*$ is not in the interior of $V(N)$. So $\bar{t} > 0$, and $v^{*i} > \bar{v}^i$ for all $i \in N$.

If $v^*$ were an element of $V(Q)$ for some $Q \in \mathcal{N}^*$, then $\bar{v}$ would be an interior point of $V(Q)$, contradicting the choice of $\bar{v}$ in core of the game $\bar{V}$. □

5. Existence results

We are now in a position to state our first existence result. If the instantaneous NTU-games are additively $b$-balanced in all states, then by Proposition 3 the evaluation games $\mathcal{G}_s$ possess non-empty cores. Consequently every subgame of the stationary cooperative game has a non-empty classical core. Thus we have proved the following theorem.

**Theorem 3.** Let $\Gamma$ be a stationary cooperative game satisfying Assumption (A), and let $b \in B$. Suppose that each of the instantaneous NTU-games $U_s$, $s \in S$, is additively $b$-balanced. Then the classical core of any subgame of $\Gamma$ is non-empty.

Let us now turn to the question of non-emptiness of the strong sequential core. Consider a family of stationary cooperative games $\Gamma_\delta$ parameterized by the discount factor $\delta \in (0, 1)$. We are interested in the behavior of the strong sequential core as $\delta$ approaches one. In the rest of this section, we use the notation $\psi_\delta$ rather than $\psi$, $v_{\delta s}$ rather than $v_s$ and $V_{\delta s}$ rather than $V_s$, to emphasize the dependence of the respective objects on the discount factor. Of course, $\psi_\delta$ denotes the matrix $(1 - \delta)(I_S - \delta \pi)^{-1}$, with $\psi_\delta(\sigma|s)$ its element in the row $\sigma$ and column $s$, $v_{\delta s}$ denotes the utility function in the evaluation game $\mathcal{G}_s$ and $V_{\delta s}$ denotes the NTU-game associated with $\mathcal{G}_s$, for a given discount factor $\delta$.

Below we state a mathematical result that plays a crucial role in the analysis of the asymptotic behavior of the strong sequential core. The proof can be found in Predtetchinski et al. (2004).

First recall that, for a given column stochastic $(S \times S)$-dimensional matrix $\pi$, the stationary distribution $\phi$ is a probability distribution over the set $S$ satisfying the equation $\pi \phi = \phi$. Here we regard $\phi$ as a column $S$-dimensional vector. As we assume (A5) the matrix $\pi$ to be irreducible, there exists a unique stationary distribution $\phi$. Moreover, $\phi(s) > 0$ for all $s \in S$.
Theorem 4. Suppose that the matrix \( \pi \) is irreducible. Let \( \phi \) be the unique stationary distribution over the set \( S \). Then for all \( \sigma \) and \( s \) in \( S \)

\[
\lim_{\delta \to 1, 0 < \delta < 1} \psi_\delta(\sigma | s) = \phi(\sigma).
\]

(3)

Definition 5. The game \( \mathcal{G}_\infty \) is a cooperative game where the possibilities of a coalition \( Q \) are represented by the set \( \times_{\sigma \in S} U_\sigma(Q) \), and the players’ expected utilities from consuming the state-contingent utility tuple \( u \in \mathbb{R}^{n_S} \) are given by the vector

\[
v_\infty(u) = \sum_{\sigma \in S} \phi(\sigma) u_\sigma.
\]

Let \( C^*(\mathcal{G}_\infty) \) denote the set of utility tuples \( u \in \times_{\sigma \in S} U_\sigma(N) \) such that there is no \( z \in \times_{\sigma \in S} U_\sigma(N) \) with \( v^i_\infty(z) > v^i_\infty(u) \) for all \( i \in N \), and no \( Q \in \mathcal{N}^* \) and \( y \in \times_{\sigma \in S} U_\sigma(Q) \) with \( v^i_\infty(y) \geq v^i_\infty(u) \) for all \( i \in Q \).

Thus the utility tuple \( u \) is an element of the set \( C^*(\mathcal{G}_\infty) \) if it is in the core of the game \( \mathcal{G}_\infty \) and its expected value \( v_\infty(u) \) cannot be attained by any proper coalition.

Associated with the game \( \mathcal{G}_\infty \) is an NTU-game \( V_\infty \) defined as follows:

\[
V_\infty(Q) = \sum_{\sigma \in S} \phi(\sigma) U_\sigma(Q).
\]

By an argument analogous to the proof of Proposition 2, \( V_\infty \) satisfies conditions (G1)–(G3), provided that the stationary cooperative game \( \Gamma \) satisfies Assumption (A). Clearly, \( u \) is an element of \( C^*(\mathcal{G}_\infty) \) if and only if \( v_\infty(u) \) is an element of \( C^*(V_\infty) \).

By Theorem 4, the utility function \( v_{\delta s} \) converges to the utility function \( v_\infty \) as \( \delta \) approaches one, in the following sense:

\[
\lim_{\delta \to 1, 0 < \delta < 1} v_{\delta s}(u) = v_\infty(u).
\]

(4)

This observation plays a central role in the proof of Theorem 5.

Theorem 5. Let \( \Gamma_{\delta} \) be a family of stationary cooperative games satisfying Assumption (A). Let \( u^* \) be an element of the set \( C^*(\mathcal{G}_\infty) \). Then there exists a \( \delta^* \in (0, 1) \) such that the stationary utility stream \( \{u^*_s\} \) is an element of the strong sequential core of the game \( \Gamma_{\delta} \) whenever \( \delta^* < \delta < 1 \).

Proof. We must show that \( u^* \) belongs to the core of the evaluation game \( \mathcal{G}_s \) for all \( s \in S \) and all \( \delta \)’s sufficiently close to one.

First we show that \( v_{\delta s}(u^*) \) lies in the boundary of \( V_{\delta s}(N) \) for all \( s \in S \) and for all \( \delta \in (0, 1) \). Since \( v_\infty(u^*) \in \partial V_\infty(N) \), there exists a non-zero vector \( \alpha \in \mathbb{R}^n \) such that \( v_\infty(u^*) \) maximizes the scalar product \( \alpha \cdot v \) over all \( v \in V_\infty(N) \). Since \( \phi(\sigma) \) is positive for all \( \sigma \in S, u^*_\sigma \) maximizes \( \alpha \cdot u_\sigma \) over all \( u_\sigma \in U_\sigma(N) \). Therefore, \( v_{\delta s}(u^*) \) is a maximizer of \( \alpha \cdot v \) over all \( v \in V_{\delta s}(N) \), for all \( s \in S \) and \( \delta \in (0, 1) \). This implies that \( v_{\delta s}(u^*) \) is a boundary point of \( V_{\delta s}(N) \).

Now we show that \( v_{\delta s}(u^*) \) is in the complement of \( V_{\delta s}(Q) \) for all \( s \in S, Q \in \mathcal{N}^* \) and for all \( \delta \) close to one. Suppose not. Then there exists a state \( s \in S \), a coalition \( Q \in \mathcal{N}^* \) and a sequence \( \delta^* \in (0, 1) \) converging to one such that \( v_{\delta^* s}(u^*) \) is an element of \( V_{\delta^* s}(Q) \) for all \( q \). Let the subset \( C \) of \( \mathbb{R}^n \times \mathbb{R}^S \) be defined as

\[
C = \left\{ \sum_{\sigma \in S} \gamma_\sigma U_\sigma(Q) \right\} \times \{\gamma_1, \ldots, \gamma_S\}.
\]
where the union is taken over all choices of non-negative coefficients $\gamma_1, \ldots, \gamma_S$ adding up to 1. By Eqs. (3) and (4), the sequence
\[
[v_{\delta q}(u^*), \psi_{\delta q}(1|s), \ldots, \psi_{\delta q}(S|s)]
\]
converges to the vector $[v_{\infty}(u^*), \phi]$. Since each vector in the sequence is an element of the set $C$, the vector $[v_{\infty}(u^*), \phi]$ belongs to the closure of $C$.

Let $\{e_1, \ldots, e_S\}$ be the standard ordered basis of $\mathbb{R}^S$, and let the subset $C_\sigma$ of $\mathbb{R}^n \times \mathbb{R}^S$ be defined as $C_\sigma = U_\sigma(Q) \times \{e_\sigma\}$. Observe that the set $C$ can be written as a union
\[
C = \bigcup \{\sum_{\sigma \in S} \gamma_\sigma C_\sigma\}.
\]
It follows from Theorem 3.3 in Rockafellar (1997, p. 18) that the set $C$ is the convex hull of the union $\bigcup_{\sigma \in S} C_\sigma$. By Theorem 9.8 in Rockafellar (1997, p. 80) and by Assumption (A4), the closure of $C$ is therefore given by
\[
\text{cl } C = \bigcup \left\{ \sum_{\sigma \in S} \gamma_\sigma C_\sigma + \sum_{\sigma \in S} \mathcal{O}^+[C_\sigma] \right\}
\]
\[
= \bigcup \left\{ \sum_{\sigma \in S} \gamma_\sigma U_\sigma(Q) + \sum_{\sigma \in S} \mathcal{O}^+[U_\sigma(Q)] \right\} \times \{\gamma_1, \ldots, \gamma_S\}.
\]
Thus, the inclusion $[v_{\infty}(u^*), \phi] \in \text{cl } C$ together with the positivity of the probabilities $\phi(\sigma)$ implies that $v_{\infty}(u^*)$ is an element of $\sum_{\sigma \in S} \phi(\sigma) U_\sigma(Q) = V_{\infty}(Q)$. However, this contradicts the choice of $u^*$ in $C^*(g_{\infty})$. $\square$

By an argument similar to the proof of Proposition 3 one can show that the NTU-game $V_{\infty}$ is strongly additively $b$-balanced, provided that the instantaneous NTU-games are additively $b$-balanced in all states, and at least one of these NTU-games is strongly additively $b$-balanced. In this case the set $C^*(g_{\infty})$ is non-empty. We have thus established the following result.

**Corollary 2.** Let $\Gamma_{\delta}$ be a family of stationary cooperative games satisfying Assumption (A), and let $b \in B$. Suppose that each of the NTU-games $U_s$, $s \in S$, is additively $b$-balanced, and at least one of these NTU-games is strongly additively $b$-balanced. Then there exists a $\delta^* \in (0, 1)$ such that the strong sequential core of the game $\Gamma_{\delta}$ is non-empty whenever $\delta^* < \delta < 1$.

### 6. An example

We construct a family of stationary cooperative games where the strong sequential core is empty for all $\delta \in (0, 1)$. In this example the stationary cooperative games satisfy Assumption (A*), and the NTU-game $U_s$ is additively $b$-balanced for all $s \in S$. However, no $U_s$ has the property of strong additive $b$-balancedness.

**Example 3.** Consider a family $\Gamma_{\delta}$ of stationary cooperative games with three players called $a$, $b$, and $c$, and two states of nature. The transitional probabilities are given by the matrix
\[
\pi = \begin{bmatrix}
0.75 & 0.25 \\
0.25 & 0.75
\end{bmatrix}.
\]


The stationary distribution over \( S \) is given by the vector \( \phi = (0.5, 0.5) \). Table 1 reports the equations defining the sets of instantaneous utility payoffs.

To prove that the assumptions (A3) and (A4) are satisfied, we show that for each coalition \( Q \) and each state \( s \) the recession cone of \( U_s(Q) \) is given by the set \( \mathbb{R}^a_+ = \{ y \in \mathbb{R}^a \mid y_i \leq 0 \ \forall \ i \in Q \} \). The inclusion \( \mathbb{R}^a_+ \subset \mathcal{O}^+[U_s(Q)] \) follows automatically from condition (G2). For the coalition \( Q = \{ a, b \} \) the converse inclusion follows from the fact that \( u^a \leq \ln 6 \) and \( u^b \leq \ln 6 \) for all \( u \in U_s((a, b)) \).

Consider now \( Q = \{ a, c \} \). Let the vector \( y \) be an element of \( \mathcal{O}^+[U_s((a, c))] \). By definition of the recession cone, \( y + u \in U_s((a, c)) \) whenever \( u \in U_s((a, c)) \). This implies that the vector \( y \) must satisfy the inequality \( \exp(u^a) \exp(y^a - 1) + y^c \leq 0 \) for all real numbers \( u^a \). Taking the limit of the left-hand side of the inequality as \( u^a \) approaches \( -\infty \) yields \( y^c \leq 0 \). Rewriting the inequality as \( \exp(y^a - 1) \leq -y^c \exp(-u^a) \) and taking the limit of the right-hand side as \( u^a \) approaches \( +\infty \) yields \( y^a \leq 0 \). Thus, \( y \in \mathbb{R}^{(a,c)} \), as desired. The cases \( Q = \{ b, c \} \) and \( Q = \{ a, b, c \} \) are dealt with similarly.

Let \( b'_Q = 1 \) for all \( Q \in \mathcal{N}^* \) and all \( i \in Q \). Using the convexity property of the exponential function it is not difficult to show that the games \( U_s \) are additively \( b \)-balanced. To see that the game \( U_1 \) violates the condition of strong additive \( b \)-balancedness for any \( b \in B \), observe that the core of \( U_1 \) consists of the vector \( (\ln 4, \ln 2, 4) \) alone, and that this vector is feasible for every coalition of size 2. Thus the set \( C^*(U_1) \) is empty. A similar argument shows that the game \( U_2 \) violates the condition of strong additive \( b \)-balancedness.

Table 2 reports the equations defining the sets \( V_{\infty}(Q) \) and the sets \( V_{\delta S}(Q) \). We use the notation

\[
\begin{align*}
\delta S((a, c)) &= \lfloor \psi_3(1|s) \times 8 + \psi_3(2|s) \times 6 \rfloor, \\
\delta S((b, c)) &= \lfloor \psi_3(1|s) \times 6 + \psi_3(2|s) \times 8 \rfloor.
\end{align*}
\]

Table 2 makes it clear that the core of the game \( V_{\infty} \) consists of a single point \((v^a, v^b, v^c) = (\ln 3, \ln 3, 4)\) being feasible for each coalition of size 2. Now a state-contingent utility tuple \((u_1, u_2)\) in \( U_1(N) \times U_2(N) \) belongs to the core of the game \( \mathcal{G}_{\infty} \) if and only if \( \frac{1}{2} u_1 + \frac{1}{2} u_2 = v \). Because \( U_1(N) = U_2(N) = V_{\infty}(N) \) and because this set is strictly convex we must have \( u_1 = u_2 = v \), for otherwise the point \( v \) would be in the interior of the set \( V_{\infty}(N) \). We see that the core of the game \( \mathcal{G}_{\infty} \) is a one-point set

\[
C(\mathcal{G}_{\infty}) = \left\{ \begin{array}{c|c|c}
(u^a_1 & u^a_2 & u^a_3 = \ln 3 \\
u^b_1 & u^b_2 & u^b_3 = \ln 3 \\
u^c_1 & u^c_2 & u^c_3 = 4
\end{array} \right\}.
\]
Table 2  
Equations defining the sets $V_{\delta s}(Q)$ and $V_{\infty}(Q)$

<table>
<thead>
<tr>
<th>$Q$</th>
<th>$V_{\delta s}(Q)$</th>
<th>$V_{\infty}(Q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>{a}</td>
<td>$\exp(v^a)$</td>
<td>$\leq 1$</td>
</tr>
<tr>
<td>{b}</td>
<td>$\exp(v^b)$</td>
<td>$\leq 1$</td>
</tr>
<tr>
<td>{c}</td>
<td>$v^c$</td>
<td>$\leq 1$</td>
</tr>
<tr>
<td>{a, b}</td>
<td>$\exp(v^a) + \exp(v^b)$</td>
<td>$\leq 6$</td>
</tr>
<tr>
<td>{a, c}</td>
<td>$\exp(v^a) + v^c$</td>
<td>$\leq w_{\delta s}([a, c])$</td>
</tr>
<tr>
<td>{b, c}</td>
<td>$\exp(v^b) + v^c$</td>
<td>$\leq w_{\delta s}([b, c])$</td>
</tr>
<tr>
<td>{a, b, c}</td>
<td>$\exp(v^a) + \exp(v^b) + v^c$</td>
<td>$\leq 10$</td>
</tr>
</tbody>
</table>

The structure of the NTU-game $V_{\delta s}$ for any $\delta \in (0,1)$ is similar to that of the game $V_{\infty}$, because $w_{\delta s}([a, c]) + w_{\delta s}([b, c]) = 14$. It follows that the core of the NTU-game $V_{\delta s}$ consists of a single point which is feasible for any coalition of size 2:

$$C(V_{\delta s}) = \left\{ \begin{pmatrix} v^a \\ v^b \\ v^c \end{pmatrix} : \begin{vmatrix} u^a_1 = u^a_2 = \ln[w_{\delta s}([a, c]) - 4] \\ u^b_1 = u^b_2 = \ln[w_{\delta s}([b, c]) - 4] \\ u^c_1 = u^c_2 = 4 \end{vmatrix} \right\}.$$ 

As the feasible set of the grand coalition is state-independent and strictly convex, any element in the core of the evaluation game is state-independent. It follows that the core of the evaluation game $\Gamma_{\delta s}$ is a singleton:

$$C(\Gamma_{\delta s}) = \left\{ \begin{pmatrix} u^a_1 = u^a_2 = \ln[w_{\delta s}([a, c]) - 4] \\ u^b_1 = u^b_2 = \ln[w_{\delta s}([b, c]) - 4] \\ u^c_1 = u^c_2 = 4 \end{pmatrix} \right\}.$$ 

Now one can check that $w_{\delta 1}([a, c]) \neq w_{\delta 2}([a, c])$ implying that the intersection $C(\Gamma_{\delta 1}) \cap C(\Gamma_{\delta 2})$ is an empty set for all $\delta \in (0, 1)$. Corollary 1 implies that the strong sequential core of the game $\Gamma_{\delta}$ is empty for all $\delta \in (0, 1)$.

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References