OVERHEAD LABOUR AND BOUNDED SUBSTITUTABILITY**

BY

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INTRODUCTION

Overhead labour is a common phenomenon nowadays in the production process. It results from tasks like management, supervision, maintenance and research. Partly due to the nature of the work itself, partly due to its high costs of hiring and training, overhead labour differs from production labour in that it is 'quasi fixed' in the short run. That is, it is less sensitive to changes in the wage rate and in output than production labour is. For that reason one can wonder whether labour still can be considered as homogeneous in the short-run production function of an industry. However, production labour and overhead labour virtually have the same price, the wage rate. Moreover, in many cases data are only available on total labour employed. Therefore we shall consider labour as homogeneous.

But then the question arises whether the properties of the short-run production function – which usually is assumed to be well-behaved – are not affected by the presence of quasi-fixed overhead labour. We shall argue below that these properties will be affected indeed: when employing overhead labour the industry will only be able to earn a positive quasi-rent for a bounded interval of the aggregate labour intensity. We call this property of the aggregate production function 'bounded substitutability.'

In order to explain this one should realise that the usual distinction between the long run, in which both capital and labour are variable, and the short run, in which only labour varies, is no longer appropriate. In section 1 we argue that one should distinguish between the very-, the rather- and the quasi short run in

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1 For the United States, 1949-1979, the annual wages per worker for production workers and nonproduction workers moved parallel: both increased by a factor 4.7 during that period. Their levels differ, however, by about 50%. Cf. Sato (1983), data appendix.
2 This property has been studied extensively in Kuipers (1970) and Kuipers and Muysken (1974). However, no reference is made to overhead labour.
order to analyse the behaviour of firms in the presence of overhead labour. We demonstrate in section 2 that both in the very- and in the rather- short run the industry's production function will show bounded substitutability. Moreover, bounded substitutability can even occur in the quasi short run, as is elaborated in section 3. Some concluding remarks are presented in section 4.

I THE BEHAVIOUR OF FIRMS WITH RESPECT TO OVERHEAD LABOUR

Let each firm in an industry consist of several production units, which employ both overhead labour and production labour. For simplicity we assume that all overhead labour of a firm is employed by its production units. Then we can consider the industry as a collection of production units.

Once a production unit has been installed, substitution between overhead labour and production labour still may be possible. This is reflected in the production function:

\[ q = ak f[\beta l_p/ak, \pi l_o/ak] \]  

where \( q \) stands for output, \( k \) stands for the given capital stock, \( l_p \) stands for production labour, and \( l_o \) stands for overhead labour. The values of the efficiency parameters \( \alpha, \beta \) and \( \pi \) differ across production units.\(^3\)

At a given wage rate, for each production unit that combination of overhead labour and production labour will be chosen from equation (1) which maximises its profits. However, once such a combination is chosen, the amount of overhead labour is quasi-fixed. That is, due to its high adjustment costs, overhead labour will not change immediately in response to a change in the wage rate. This situation is characteristic of the **quasi short run**.

The chosen combination of overhead labour and production labour in the quasi short run can be represented by:

\[ l_o = \mu l_p \]  

In this equation \( l_p \) stands for the amount of production labour employed when the production unit is operating at full capacity in the quasi short run. It is fixed as long as overhead labour is. Obviously, the value of \( \mu \) will differ across production units. In this situation output is produced according to:

\[ q = ak g[\beta l_p/ak] \]  

where \( g \) is assumed to be well-behaved.\(^4\) At a given wage rate, production units

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\(^3\) The values of \( \pi \) can be corrected for differences between wages for overhead labour, \( w_o \), and production labour, \( w_p \), such that \( \pi = w_o \pi^*/w_p \), where \( \pi^* \) stands for the true value of \( \pi \).

\(^4\) See appendix 1.
now will choose the amount of production labour according to equation (3)
such that profits are maximized, given overhead labour and capital. When in
that situation quasi-rent over production labour is negative, production units
will not employ production labour and not produce any output. They will be
able to keep their overhead labour employed only for a very limited amount of
time: the very short run.

After this very short run only those production units that at least have a
positive quasi-rent over production labour can survive and employ overhead
labour. This is characteristic for the rather short run. Finally, however, only
those production units will remain viable that have a positive quasi-rent over
total labour employed. The other units have to revise their amount of overhead
labour employed, which brings us back to the quasi short run.\textsuperscript{4a}

\section*{2 THE VERY AND THE RATHER SHORT RUN}

In the very and the rather short run each production unit produces output
according to equation (3) and employs overhead labour according to equation
(2). It can be characterised by its capital stock, measured by ak, and its
parameters, \( \beta \) and \( \mu \). The parameters \( \beta \) and \( \mu \) have maximum values, \( b \) and \( u \)
respectively. Let the capital stock of the production units be distributed accord-
ing to \( W(\beta, \mu) \). We assume\textsuperscript{5}:

\[ W(\beta, \mu) = \frac{1}{b u} \Omega \left( \frac{\beta \mu}{b u} \right) \quad \beta < b, \mu < u \]  

(4)

where \( \Omega \) is a density function and \( K \) is some measure of the aggregate capital
stock.\textsuperscript{6} Obviously, capacity output of the industry, \( \bar{Q} \), is equal to \( aK \).

Since both in the very and in the rather short run only those production units
are producing output that have a positive quasi-rent over production labour,
the relation between aggregate output, \( Q \), and production labour, \( L_p \), can be
derived according to the distribution approach.\textsuperscript{7} The relation is given by the
well-behaved short-run production function:

\[ Q = aK F \left( \frac{bL_p}{aK} \right) \]  

(5)

with properties \( F(0) = 0, F(C) = 1, F'(0) = 1 \) and \( F'(C) = 0: C = bL_p/aK \) is a
constant. We are looking, however, for the relation between aggregate output

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\textsuperscript{4a} The introduction of adjustment costs for overhead labour makes it possible to analyse the
transition between the several kinds of short run.

\textsuperscript{5} This form of \( \Omega \) is necessary to guarantee that the form of \( F \) in (5) does not change over time.

\textsuperscript{6} Cf. Sato (1975), pp. 132-133.

\textsuperscript{7} Cf. Johansen (1972) and Sato (1975). Equation (5) is derived in appendix 1.
and aggregate labour, \( L = L_p + L_o \), where \( L_o \) stands for aggregate overhead labour.

In the very short run aggregate overhead labour will be proportional to full-capacity production labour, \( \bar{L}_p \):

\[
L_o = \delta \bar{L}_p
\]

since all production units in the industry will keep their overhead labour employed. As a consequence, the industry's short-run production function is:

\[
Q = aK.F(\frac{bL}{aK} - \delta.C)
\]

A remarkable property of this function is that output can only be positive when aggregate labour exceeds some positive amount.\(^8\) From the above it is obvious that output can only be positive in the very short run when total aggregate labour exceeds fixed aggregate overhead labour. Moreover:

\[
\frac{dQ}{dL_p} = \frac{1}{\frac{dQ}{dL}} = x
\]

holds, where \( x \) stands for the wage rate in terms of output price. Then the industry's quasi-rent will only be positive when aggregate labour exceeds some positive amount \( L^* > L_o > 0 \), where \( L^* \) is found from:

\[
F(\frac{bL^*}{aK} - \delta.C) = \frac{bL^*}{aK}.F'(\frac{bL^*}{aK} - \delta.C)
\]

Hence, in order to earn a positive quasi-rent, aggregate labour employed has to exceed aggregate overhead labour significantly, since first sufficient production labour has to be employed to cover the costs of overhead labour. Therefore it is obvious that the production function (7) has the property of bounded substitutability.

In the rather short run one finds:

\[
L_o = \zeta(Q/\bar{Q}).L_p
\]

Because those production units whose quasi-rent over production labour is negative are laid off, overhead labour is proportional to actual instead of full-capacity production labour of the industry. Actually the factor \( \zeta(Q/\bar{Q}) \) depends, via \( x \), on \( Q/\bar{Q} \), since not each production unit will be producing output

\(^8\) An example of such a function is: \( Q/K = a_1L/K + a_2(L/K)^2 + a_3 \). Cf. Kuipers (1970). This function is derived in the context of the distribution approach in Muysken (1983).
at full capacity and the μ's will differ across units. For the sake of exposition, however, we assume a fixed coefficient production function for (3) and identical values of μ for each production unit. Then $\zeta(Q/\bar{Q}) = \zeta$, which is independent of other variables. The case of a general production function, and different values for μ, leads to similar results. It is presented in appendix 1.

It is obvious that the aggregate production function in the rather short run becomes:

$$Q = aK.F\left( \frac{bL}{aK.(1+\xi)} \right)$$  \hspace{1cm} (11)

Contrary to the very short run, overhead labour will no longer set a lower limit to the amount of labour necessary to produce any output at all. However, since:

$$\frac{dQ}{dL} = \frac{x}{1+\xi}$$  \hspace{1cm} (12)

holds, the industry's quasi-rent will only be positive when aggregate labour exceeds $L^{**}$. This can be found from:

$$F\left( \frac{bL^{**}}{aK.(1+\xi)} \right) = \frac{bL^{**}}{aK} \cdot F'(\frac{bL^{**}}{aK.(1+\xi)})$$  \hspace{1cm} (13)

It is obvious that $L^{**}$ has to exceed the overhead labour employed. Hence, as is the case in the very short run, the aggregate production function (13) has the property of bounded substitutability: the industry can earn a positive quasi-rent only over a limited interval of the aggregate labour intensity. However, the interval will be wider than in the very short run.

3 THE QUASI SHORT RUN

In the quasi short run each production unit produces according to equation (1). It can be characterised by its given capital stock, measured by $ak$, and its efficiency parameters $\beta$ and $\pi$. Let the capital stock of the production units be distributed according to $V(\beta, \pi)$. We assume, analogous to equation (4):

$$V(\beta, \pi) = a.K.\frac{1}{b}.\frac{1}{p}.\Gamma\left(\frac{\beta}{b}, \frac{\pi}{p}\right) \quad \beta < b, \pi < p$$  \hspace{1cm} (14)

where $\Gamma$ is a density function.

9 From $F(z) > z.F'(z)$ for $z > 0$ and $F'(0) = 1$ it follows that $L_p^{**} > 0$ for $\zeta > 0$. 
At a given wage rate production units will choose the combination of production labour and overhead labour that maximises their profits, given the capital stock. Those units that can earn a positive quasi-rent will actually employ labour and produce output. As is explained in appendix 2, this results in the following aggregate production function:

\[ Q = aK \cdot G\left(\frac{bL_p}{aK}, \frac{pL_o}{aK}\right) \]  

(15)

The properties of this function will depend on those of the production function (1) of the production units. Obviously the marginal productivity conditions will hold, i.e. \( b \cdot G_1 = p \cdot G_2 = x \). Moreover, diminishing returns will be present with respect to both productive labour and overhead labour, i.e. \( G_{11} < 0 \) and \( G_{22} < 0 \), assuming diminishing returns on the unit level, i.e. \( f_{11} < 0 \) and \( f_{22} < 0 \). Furthermore, since overhead labour reacts more sluggishly to changes in the real wage rate, \( G_{22} < G_{11} \) holds, assuming that production labour exceeds overhead labour. A property of crucial importance, finally, is the reaction of the marginal productivity of production labour to an increase in overhead labour, i.e. the sign of \( G_{12} \).

When the marginal productivity of production labour increases with overhead labour at the unit level, the aggregate production function can show increasing returns to aggregate labour. The explanation of this phenomenon is that when the wage rate goes down, less efficient production units that had been laid off start to operate again. At the same time the units that already were active will operate at a higher level of capacity utilisation and increase both their production labour and overhead labour employed. The increase in both kinds of labour will lead to decreasing returns in overhead labour and production labour separately. However, the combined increase can at the same time lead to increasing returns in the case of a positive effect of the increase in overhead labour on the marginal productivity of production labour. This might even offset the separate effects of diminishing returns such that finally increasing returns to aggregate labour prevail.

Up to this point we distinguished between production labour and overhead labour in the aggregate production function (15). However, the aggregate production function can also be written as:

\[ Q = aK \cdot H\left(\frac{eL}{aK}\right) \]  

(16)

11 Actually this will be the case when \( G_{12}^2 > G_{11} G_{22} \). See also Sato (1983), p. 6. One should realise, however, that a stable equilibrium never can be found here in a situation of perfect competition. Cf. Borts and Mishan (1962), pp. 307-308. Sato (1983) does not have this problem, since he assumes imperfect competition.
13 See appendix 2.
Obviously \( e. H' = x \) holds. We also argued above that \( H'' > 0 \) is possible. In that case an increase in the amount of labour employed will be accompanied by a decrease in the quasi-rent. The quasi-rent cannot become negative, however, since only those production units will be operating that have a positive quasi-rent. Moreover, since the marginal cost curve is decreasing, there cannot be a stable equilibrium.

It seems reasonable to assume that the mutual benefits of production labour and overhead will only outweigh their separate decreasing returns over a limited interval, say \( l = l_p + l_o < l' \), in the production function (1).\(^{14}\) Then obviously the aggregate production function (16) will also have increasing returns to labour over a limited interval only, say \( L < L' \). However, when \( l < l' \), an increase in the amount of labour employed by a production unit will be accompanied by an increase in quasi-rent. It then is very well possible that quasi-rent is negative, say for \( l < l^0 \), and production units will only start operating with \( l > l^0 \).\(^{15}\) As a consequence, the aggregate production function (16) will only be defined over a limited interval of aggregate labour, say for \( L > L^0 \). It is obvious that, in that case, the aggregate production function (16) also has the property of bounded substitutability.

4 CONCLUDING REMARKS

We argue in this paper that the presence of quasi-fixed overhead labour affects the properties of the short-run production function of an industry. Instead of being well-behaved, as is usually assumed, the aggregate production function has the property of bounded substitutability. That is, the industry can only earn a positive quasi-rent for a bounded interval of the aggregate labour intensity.

The introduction of overhead labour into the analysis also makes the familiar distinction between the short run and the long run somewhat ambiguous. Actually one should distinguish between several types of the short run.

In the very short run each production unit employs a fixed amount of overhead labour. Moreover, even units that cannot earn a positive quasi-rent over their production labour will try to survive. As a consequence, all production units keep their overhead labour employed, and aggregate overhead labour is fixed. It is obvious that this causes bounded substitutability.

In the rather short run, only those production units employ overhead labour that at least earn a positive quasi-rent over their production labour. Then aggregate overhead labour is no longer fixed, although it still is fixed at the unit

\(^{14}\) Hence \( f_{z} > f_{\delta} \) for \( l < l', l' \) will differ across production units. An example of such a function is:

\[
q = c \cdot l_p / k - d \cdot k \cdot (l_p / k)^\gamma \cdot (l_o / k)^\delta \\
\gamma > \delta > 1, c, d > 0.
\]

\(^{15}\) It is obvious that \( l^0 > l' ; l^0 \) will also differ across production units. We ignore the possibility that the quasi-rent is positive for \( 0 < l < l_{\min} < l' \).
A. The very short run, equation (7)

B. The rather short run, equation (11)

C. The quasi short run, equation (16)

Figure 1 - Overhead labour and the form of the production function
level. Hence, since each production unit employs a fixed amount of overhead labour, bounded substitutability still occurs.

Finally, in the quasi short run, production units can vary the amount of overhead labour employed. Then they can benefit from the positive effect of an increase in overhead labour on the marginal productivity of production labour. This effect is limited, however, since the capital stock is fixed. Nonetheless, it is possible that initially this effect causes increasing returns to labour, even such that the quasi-rent of a production unit is negative for low levels of employment. In that case production units will only start operating at higher levels of employment. Then bounded substitutability will also be observed in the rather short run.

The above cases are summarised in Fig. 1, where $L^*$ indicates the minimum amount of labour that has to employed in the industry in order to earn a positive quasi-rent. One sees that in all forms of the short run, the presence of overhead labour can explain the phenomenon of bounded substitutability in the production function of the industry.

**APPENDICES**

*Appendix 1 Aggregation in the very and the rather short run*

We assume that the production function (3):

$$ q = \alpha k g(\beta l_p/\alpha k) $$

(A.1.1)

has the properties $g(0) = 1, g'(c) = 1, g'(0) = 1$ and $g'(c) = 0$. This implies that $l_p = \alpha k c/\beta$. Define the function $h$ such that $g'(c) = 0 \rightarrow h(0) = c$. Then we find for the very short run, according to the distribution approach:

$$ \frac{Q}{\bar{Q}} = \int \int g \left( h\left(\frac{x/b}{z}\right) \right) .\Omega(z, y).dy.dz = m(x/b) $$

(A.1.2)

$$ \frac{L_p}{\bar{Q}} = \frac{1}{b} \int \int \frac{1}{z} h\left(\frac{x/b}{z}\right) .\Omega(z, y).dy.dz = \frac{1}{b} n(x/b) $$

(A.1.3)

$$ \frac{L_o}{\bar{Q}} = \frac{1}{b} \int \int \frac{y}{z} .\Omega(z, y).dy.dz = \frac{1}{b} \delta C $$

(A.1.4)\(^{16}\)

Elimination of $x/b$ from equations (A.1.2) and (A.1.3) gives the aggregate short-run production function (5):

\(^{16}\) Equation (A.1.4) implicitly defines $\delta$.  

\[ Q = aK.F(bL_p/aK) \]  
\hspace{0.5cm} \text{(A.1.5)}

whereas from (A.1.3) one finds:
\[ \bar{L}_p = n(0)\bar{Q}/b = C.\bar{Q}/b \]  
\hspace{0.5cm} \text{(A.1.6)}

Substitution of (A.1.6) in (A.1.4) finally gives equation (6):
\[ L_o = \delta \bar{L}_p \]  
\hspace{0.5cm} \text{(A.1.7)}

In the rather short run, equation (A.1.4) should be replaced by:
\[ \frac{L_o}{\bar{Q}} = \frac{u}{b} \int_0^{1} \int_{x/b}^{1} \frac{1}{z} c \Omega(z, y) dy dz = \frac{1}{b} p(x/b) \]  
\hspace{0.5cm} \text{(A.1.8)}

Since equations (A.1.2) and (A.1.3) do not change, the aggregate production function (A.1.5) still holds. Elimination of \( x/b \) from equations (A.1.2), (A.1.3) and (A.1.8) yields equation (10):
\[ L_o = \zeta(\bar{Q}/\bar{Q})L_p \]  
\hspace{0.5cm} \text{(A.1.9)}

where \( \zeta(\omega) = n\{m^{-1}(\omega)/p\{m^{-1}(\omega)} \} \).

The industry will only earn a positive quasi-rent when aggregate labour exceeds \( L^{**} \). This can be found from:
\[ L^{**} = L_p^{**} \{ 1 + e \{ F(bL_p^{**}/aK) \} \} \]  
\hspace{0.5cm} \text{(A.1.10)}

while \( L_p^{**} = t.a.K/b \), where \( t \) satisfies:
\[ F(t) = t.F'(t) \{ 1 + e \{ F(t) \} \} \]  
\hspace{0.5cm} \text{(A.1.11)}

Substitution of equation (A.1.10) in (A.1.11) yields the equivalent of equation (13).

Appendix 2 Aggregation in the quasi short run

Since \( L_p \) and \( L_o \) appear as separate variables in the production function (1), with different efficiency parameters, we treat them initially as different inputs, with prices \( x_o \) and \( x_p \), respectively. Using the capacity distribution (14) and the constraint of a positive quasi-rent for individual firms, it can be derived that the aggregate output, production labour and overhead labour are functions of the input prices, similar to equations (A.1.2) and (A.1.3)\(^{17} \):

\(^{17}\) Cf. Sato (1975), pp. 65–67. Johansen (1972) discusses the case of two variable inputs extensively. His analysis is not appropriate here, however, since he assumes fixed input coefficients, i.e. in (1) \( f_{12} = 0 \). In order to avoid complications we assume here \( f_{12} < f_{11}f_{22} \).
\[ Q = aK.m(x_o, x_p) \]  
\[ bL_p = aK.n_p(x_o, x_p) \] 
\[ pL_o = aK.n_o(x_o, x_p) \]

Elimination of \( x_o \) and \( x_p \) from (A.2.1) yields:

\[ Q = aK.G(\frac{bL_p}{aK}, \frac{pL_o}{aK}) \]  
(A.2.2)

which is equation (15). However, since holds \( x = x_p = x_o \), we can rewrite (A.2.1) as follows:

\[ Q = aK.r(x) = aK.m(x, x) \] 
\[ eL = aK.s(x) = aK.e \left\{ \frac{n_p(x, x)}{b} + \frac{n_o(x, x)}{p} \right\} \]  
(A.2.3)

Elimination of \( x \) from (A.2.3) then yields:

\[ Q = aK.H(eL/aK) = aK.r \left[ s^{-1}(eL/aK) \right] \]  
(A.2.4)

which is equation (16).

REFERENCES


Johansen, L., 1972, Production Functions, Amsterdam.


Sato, K., 1975, Production Functions and Aggregation, Amsterdam.


Summary

OVERHEAD LABOUR AND BOUNDED SUBSTITUTABILITY

In this paper we argue that due to the presence of quasi-fixed overhead labour, the aggregate short-run production function of the industry has the property of bounded substitutability. That is, when employing overhead labour, the industry will only be able to earn a positive quasi-rent for a bounded interval of aggregate labour intensity.