Cross-sectional versus time series estimation of term structure models: empirical results for the Dutch bond market

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Abstract

We compare time series and cross-section estimates of the well-known Vasicek and CIR term structure models for a dataset of daily bond prices and short-term interest rates for the Netherlands. The main finding is the great similarity between the two models with respect to the cross-sectional term structure parameter estimates and implied bond option values. From a time series perspective, we find that for some maturities the data reject the constant volatility Vasicek model and indicate the presence of the CIR volatility effects.

Keywords: Term structure; Interest rates; Bond options

JEL classification: B43, C52

1. Introduction

The valuation of interest rate contingent claims is considered to be an important area of financial research, both by academics as well as practitioners. During the
past years a lot of models have been put forward to describe the term structure and its likely movement over time. The most common approach to model the term structure of interest rates starts with the specification of a time series process to describe the behaviour of the short-term interest rate over time. Once a time series process for the short-term interest has been specified, one can work out the time series processes for bonds of all maturities, and determine the implied term structure at every time period. This can be done either in discrete time or in continuous time.\footnote{An advantage of the continuous time models is that they usually provide an analytically tractable way of dealing with the nonlinear transformations between prices and yields. In the often used class of so-called single-factor models one starts by positing a continuous time model of the instantaneous spot rate of interest. The key parameters in these models are the degree of mean reversion and the volatility of the spot rate. Some well-known models in this class are Merton (1973), Vasicek (1977), and Cox et al. (1985, CIR).} The empirical problem is to estimate the key parameters of the various models. The usual procedure is time series estimation. Chan et al. (1992) provide a recent empirical overview of the results of time series methods. The alternative pursued in this paper is cross-sectional estimation using bond prices at a single instant in time. Both procedures have their merits and drawbacks. Estimation of a mean reversion parameter for the short-term interest rate will be very difficult with time series data. One of the conclusions in the literature on unit root econometrics is that one requires many observations spanning a large number of years in order to estimate the mean reversion parameter in even the simplest model with only first-order dynamics.\footnote{But a long time series creates its own problems, since the empirical model has to cope with structural breaks (Hamilton, 1988), and other more general forms of time varying parameters. Second, a time series model that fits the short-term interest rate well, does not necessarily provide a good fit for the yield curve (see for instance the comment of Driffill (1992) on the Hamilton model).} One response to the drawbacks of the time series approach has been the development of models that start from an exogenously given term structure at a given date. Given bond prices at some date the evolution of the term structure over time is described by one or several stochastic factors. This approach was first developed by Ho and Lee (1986), and later extended by Heath et al. (1992).

Cox et al. (1985) and Longstaff and Schwartz (1992), however, argue that the
equilibrium approach is preferable to the arbitrage approach, as it is theoretically more appealing and easier to interpret. In this paper we follow the equilibrium approach, although our estimation procedure is close to the spirit of the Heath-Jarrow-Morton exogenous term structure models, in the sense that the estimation concentrates on fitting the yield curve directly for every day. We will estimate the Vasicek and CIR models on cross-sectional daily bond price data for the Netherlands, using daily data for the period 1989–1990. Our estimation procedure is closely related to Brown and Dybvig (1986), who presented empirical results for the U.S. market. ³

Cross-sectional estimation has two advantages, since the estimated model will provide the best possible fit for the term structure, and because we do not need more data than just for one day, if enough different bonds are traded on the market. However, some of the drawbacks are already apparent in the Brown and Dybvig (1986) paper, and also show up in other empirical papers. One problem is that the parameter estimates are often nonsensical (negative variances, explosive models, etc.). Another problem is that the parameters are not very stable over time.

Ideally, if the simple one-factor models of the term structure were true, the differences between time series and cross-sectional estimates should be small. But the univariate first-order autoregressive time series process that underlies the one-factor models is much too simple for the time series data. Similarly, the cross-sectional implications of the one-factor equilibrium term structure models are strongly rejected, if only because we never observe a deterministic relation between yields on bonds with different maturities. But despite being statistically rejected, the different term structure models might still do a good job in fitting the term structure and in pricing contingent claims.

If the differences between cross-sectional and time series estimation were primarily due to sampling error in the estimation, the solution could be to employ a panel data estimator, combining the information in the cross-section and time series data. In the empirical part of this paper we partly follow this idea. As will be explained in Section 4, pooling the cross-sectional data for a time series of trading days amounts to treating the spot rate as a fixed effect. ⁴

In the paper we mainly concentrate on the cross-sectional performance of the Vasicek and CIR one-factor models, since these models provide a simple closed form solution for the yield curve and for (European) option prices. Chan et al.

³ The number of studies based on cross-sectional estimation was very limited, but is growing rapidly. Brown and Dybvig (1986) is the first study for the United States. For the U.K. there is an extensive empirical study by Brown and Schaefer (1994), who consider the term structure of index bonds. Gourieroux and Scaillet (1994) estimate cross-sectional term structures on French data.

⁴ See Judge et al. (1985) for an introduction to the terminology, definitions and methods of panel data. A full treatment of the panel data methods for estimating term structure models raises additional econometric issues and is outside the scope of the present paper. Schotman (1994) deals with some of the issues.
(1991, 1992) recently investigated the time series properties of several one-factor models. Their main result is the high sensitivity of the volatility of the term structure to the level of the short rate, implying that from a time series perspective we would expect that the CIR model performs better than the simpler Vasicek model. The ranking of models from a cross-sectional perspective can be very different, though. If the time series estimates provide a poor fit for the cross-sectional yield curve, we can expect to obtain very different parameter estimates when the models are estimated on cross-section data. Although our emphasis is on cross-sectional estimation of the term structure, we also present time series parameter estimates for daily AIBOR rates in order to make our results comparable to other studies, and to appreciate the differences between time series and cross-sectional estimation.

The paper is organized as follows. Section 2 presents a short overview of the term structure models, and discusses both time series and cross-sectional estimation methods. Section 3 describes the stylized facts of the Dutch government bond data. Section 4 presents and discusses the empirical results. The paper ends with a summary and conclusions.

2. Model description

2.1. The Vasicek model

As proposed by Vasicek (1977), a possible description of the instantaneous spot risk-free rate \( r(t) \) is the Ornstein-Uhlenbeck process:

\[
    dr(t) = \kappa [\theta - r(t)] dt + \sigma dZ(t)
\]

where \( \kappa > 0 \) is the mean reversion parameter, \( \theta \) is the unconditional mean of \( r(t) \), \( \sigma \) the volatility of the spot rate, and \( Z(t) \) a standard Brownian motion. The conditional expectation and variance of the normally distributed process at time \( t + s \) given the information at time \( t \) are (see Vasicek, 1977, p. 185):

\[
    E_t[r(t+s)] = \theta + [r(t) - \theta] e^{-\kappa s}
\]

\[
    V_t[r(t+s)] = \frac{\sigma^2}{2\kappa} (1 - e^{-2\kappa s})
\]

As can be seen from these expressions, the instantaneous drift term with \( \kappa > 0 \) forces the spot rate \( r(t) \) towards its long-term mean \( \theta \). In this one-factor model, the values of bonds and options on bonds are solely a function of the short rate and claim specific characteristics like time-to-maturity. As derived in Harrison and Kreps (1979) and extended by Harrison and Pliska (1981), the exclusion of arbitrage opportunities requires the value of a discount bond \( P_t[r(t)] \) at time \( t \)
with maturity $\tau$ and face value 1 to be the expected discount factor in a risk-neutral economy:

$$ P_\tau[r(t)] = \bar{E}_t \left[ \exp \left( - \int_t^{t+\tau} r(s) \, ds \right) \right] \tag{4} $$

The expectation $\bar{E}_t$ has to be taken with respect to the risk-neutral stochastic process of the spot rate:

$$ dr(t) = \kappa \left[ \bar{\theta} - r(t) \right] dt + \sigma \, d\bar{Z}(t) \tag{5} $$

where $\bar{\theta} = \theta + \lambda$ with $\lambda$ the market price of risk and where $\bar{Z}(t)$ is another standard Brownian motion. Calculating the expectation yields the following closed form solution of the discount bond value:

$$ P_\tau[r(t)] = \exp \left( \frac{1 - e^{-\kappa \tau}}{\kappa} \left[ R_\infty - r(t) \right] - \tau R_\infty - \frac{(1 - e^{-\kappa \tau})^2}{4\kappa^3} \sigma^2 \right) \tag{6} $$

The yield-to-maturity of a discount bond is defined as $R_\tau(t) = -\ln P_\tau[r(t)]/\tau$, which implies:

$$ R_\tau(t) = R_\infty + \frac{1 - e^{-\kappa \tau}}{\kappa \tau} \left[ r(t) - R_\infty \right] + \sigma^2 \frac{(1 - e^{-\kappa \tau})^2}{4\kappa^3 \tau} \tag{7} $$

As the maturity $\tau \to \infty$, the yield-to-maturity converges to:

$$ \lim_{\tau \to \infty} R_\tau(t) = R_\infty = \bar{\theta} - \frac{\sigma^2}{2\kappa^2}, $$

which defines the symbol $R_\infty$ used in Eqs. (6) and (7).\textsuperscript{5} The properties of the Vasicek model are best understood by rewriting the yield as:

$$ R_\tau(t) = w_1 r(t) + (1 - w_1) R_\infty + w_2 \sigma^2, \tag{8} $$

where the weight functions $w_1(\kappa, \tau)$ and $w_2(\kappa, \tau)$ follow from Eq. (7) and satisfy:

$$ \lim_{\tau \to 0} w_1(\kappa, \tau) = 1, \quad \lim_{\tau \to \infty} w_1(\kappa, \tau) = 0, \quad 0 \leq w_1 \leq 1 \tag{9} $$

and

$$ \lim_{\tau \to 0} w_2(\kappa, \tau) = 0, \quad \lim_{\tau \to \infty} w_2(\kappa, \tau) = 0, \quad 0 \leq w_2 $$

\textsuperscript{5} Although in the paper we concentrate on yields, it is worthwhile to briefly review the difference between the instantaneous forward rate $f(t, t+s)$ and the expected future spot rate $r(t+s)$. In case the local expectations holds, meaning that the price of risk $\lambda$ equals zero, forward rates are downward biased estimators of expected spot rates due to Jensen's inequality. The exact functional form of the premium can be derived straightforwardly. See Ingersoll (1987) for an overview.
The yield-to-maturity is a linear combination of the risk-free rate and "infinite maturity" yield with weights summing to one, and a term that is linear in the volatility $\sigma^2$ that causes the curvature in the term structure. The effect of volatility on the yield curve is zero on both ends. The function $\omega_2$ is very sensitive to the mean reversion parameter. For large $\kappa$ it will be small for all $\tau$ and volatility effects are negligible at all maturities; for small $\kappa$ volatility effects dominate. If $\omega_2$ is small the yields are a simple weighted average of the risk-free rate and the infinite maturity yield. The smaller the value of $\kappa$, the larger will be the weight $\omega_1$ of the risk-free rate in Eq. (8).

In a cross-section the yield on a discount bond depends on four unknown parameters: $r(t)$, $\kappa$, $\sigma^2$, and $R_x$. These are sufficient to price bond options. The unconditional mean $\theta$ is not identified. A serious drawback of the Vasicek model is the possibility of negative long-term interest rates. The purpose of the parameterization (7) is that positive interest rates at all maturities can be guaranteed by the simple parameter restrictions $\kappa > 0$, $R_x > 0$, $\sigma > 0$ and $r(t) > 0$.

2.2. The Cox, Ingersoll and Ross model

As a theoretically more appealing way to overcome the negative interest rate problem Cox et al. (1985) propose the modified process:

$$dr(t) = \kappa[\theta - r(t)] \, dt + \sigma \sqrt{r(t)} \, dZ(t)$$  \hspace{1cm} (11)

If $2\kappa \theta > \sigma^2$ the upward drift term is sufficiently large to make the origin inaccessible, thereby excluding negative spot rates. The expectation of the process at time $t+s$ given the information at time $t$ is the same as for the Vasicek model. The conditional variance is given by (see Cox et al., 1985, p. 392):

$$V_t[r(t+s)] = \sigma^2 \frac{1 - e^{-\kappa s}}{2\kappa} \left[ 2r(t) e^{-\kappa s} + \theta(1 - e^{-\kappa s}) \right]$$  \hspace{1cm} (12)

Analogous to the Vasicek model, the instantaneous drift term forces the spot rate $r(t)$ towards its long-term mean $\theta$. The conditional variance, however, depends positively on the value of the spot rate. The value of a discount bond can be derived similarly as in the Vasicek model. The resulting expression for the value of a discount bond is:

$$P_\tau[r(t)] = A(\tau)^{R_x} \, e^{-B(\tau)r(t)}$$  \hspace{1cm} (13)

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6 As noted in Brown and Dybvig (1986) this observation is similar to the situation that one can only identify the parameters of the risk neutral stock returns when using option prices, and assuming that the Black-Scholes (1973) option pricing formula is correct.
with

\[ A(\tau) = \left[ \frac{2\gamma e^{(\bar{\kappa} + \gamma)\tau/2}}{(\bar{\kappa} + \gamma)(e^{\gamma\tau} - 1) + 2\gamma} \right]^\frac{\bar{\kappa} + \gamma}{\sigma^2} \]

\[ B(\tau) = \frac{2(e^{\gamma\tau} - 1)}{(\bar{\kappa} + \gamma)(e^{\gamma\tau} - 1) + 2\gamma} \]

\[ \gamma = \sqrt{\bar{\kappa}^2 + 2\sigma^2} \]

\[ R_\infty = \frac{2\kappa\theta}{\bar{\kappa} + \gamma} \]

\[ \bar{\kappa} = \kappa - \lambda \]

The notation differs slightly from the usual representation of the CIR model in order to facilitate comparison with the Vasicek model. The yield-to-maturity can be expressed as a linear function of \( r(t) \) and \( R_\infty \):

\[ R_\tau(t) = v_1 r(t) + v_2 R_\infty \]  \hspace{1cm} (14)

The weight functions \( v_1(\bar{\kappa}, \sigma, \tau) \) and \( v_2(\bar{\kappa}, \sigma, \tau) \) follow from (13) as:

\[ v_1(\bar{\kappa}, \sigma, \tau) = \frac{B(\tau)}{\tau} \]

and

\[ v_2(\bar{\kappa}, \sigma, \tau) = -\frac{\ln A(\tau)}{\tau} \]  \hspace{1cm} (15)

The weights do not sum exactly to one; they have the properties:

\[ \lim_{\tau \to 0} v_1(\tau) = 1, \quad \lim_{\tau \to \infty} v_1(\tau) = 0 \quad v_1(\tau) > 0 \]  \hspace{1cm} (16)

\[ \lim_{\tau \to 0} v_2(\tau) = 0, \quad \lim_{\tau \to \infty} v_2(\tau) = 1 \quad v_2(\tau) > 0 \]  \hspace{1cm} (17)

Like for the Vasicek model, the larger the value of \( \bar{\kappa} \), the larger the weight of the risk-free rate, and the smaller the weight of \( R_\infty \) for any given \( \tau \). The identifiable parameters for the CIR model are \( r(t) \), \( \bar{\kappa} \), \( \sigma^2 \), and \( R_\infty \). The mean reversion parameter \( \kappa \) and the unconditional mean of the spot rate \( \theta \) are not separately identified.

2.3. Time series estimation

For time series estimation we need a discrete time representation of the spot rate process. A complication is that the true instantaneous risk-free rate is not observable. The shortest available maturity is a one-month interest rate. Since the yield on all discount bonds is a simple linear function of the instantaneous
risk-free rate, the time series process of the one-month rate can be derived from the discrete time process of the instantaneous spot rate. Eqs. (2) and (3) give the moments of the discrete time process for the instantaneous risk-free rate in the Vasicek model. Since \( r(t) \) is unobservable we solve (7) for \( r(t) \) as a function of \( R_s(t) \) and substitute in (2) and (3). The final result is a discrete time process for \( R_s(t) \):

\[
R_s(t + s) = b_0 + b_1[R_s(t) - b_0] + \eta_r(t, t + s) \tag{18}
\]

with \( s \) the time between successive observations, and:

\[
b_0 = R_\infty + \frac{1 - e^{-\kappa \tau}}{\kappa \tau}(\theta - R_\infty) + \sigma^2 \frac{(1 - e^{-\kappa \tau})^2}{4 \kappa^3 \tau} \]

\[
b_1 = e^{-\kappa s}
\]

\[
E_t[\eta_r(t, t + s)^2] = s_r^2 = \sigma^2 \omega_1(\kappa, \tau)^2 \frac{1 - e^{-2 \kappa s}}{2 \kappa}
\]

Eq. (18) defines a stationary first-order autoregression with constant variance, and can be estimated by Ordinary Least Squares (OLS). The relation between \( s_r^2 \) and \( \tau \) is known as the term structure of interest rate volatilities. Not all parameters of the Vasicek model can be identified. The identifiable parameters are \( \kappa \) and \( \sigma^2 \), which can be easily retrieved from the linear regression parameters \( b_1 \) and \( s^2 \).

Neither the unconditional mean \( \theta \), nor the infinite maturity yield \( R_\infty \) can be identified. Therefore the time series estimates are not sufficient to construct cross-sectional term structures or to price bond options.

The analogous derivation for the CIR model gives

\[
R_s(t + s) = b_0 + b_1[R_s(t) - b_0] + \zeta(t, t + s) \tag{19}
\]

with \( b_1 \) as in the Vasicek model and the unconditional mean

\[
b_0 = -\ln A(\tau)\frac{R_\infty}{\tau} + \frac{B(\tau)}{\tau} \theta
\]

The variance of the error term \( \zeta(t,t+s) \) is given by

\[
E_t[\zeta(t,t+s)^2] = a_0 + aR_s(t) \tag{20}
\]

with

\[
a_0 = \sigma^2 \frac{1 - e^{-\kappa s}}{2 \kappa} \left[ \frac{B(\tau)}{\tau} \right]^2 \left[ \theta(1 - e^{-\kappa s}) + R_\infty \frac{2 \ln A(\tau) - e^{-\kappa s}}{B(\tau)} \right]
\]

\[
a_1 = \sigma^2 e^{-\kappa s} \frac{B(\tau)}{\tau} \frac{1 - e^{-\kappa s}}{\kappa}
\]
Contrary to the Vasicek model the four CIR time series parameters \((a_0, a_1, b_0,\) and \(b_1)\) enable the separate identification of the four structural parameters: \(\kappa, \sigma^2, \theta,\) and \(R_\infty.\)

In an unrestricted time series estimation the only difference between the Vasicek and the CIR model is the term \(a_1 R_\tau(t)\) in the variance function for the CIR model. For general constants \(a_0\) and \(a_1\) the above specification is more general than the Vasicek model. However, the constants depend on the structural parameters of the CIR model. For example, under the CIR specification \(a_1\) can never be zero without forcing \(a_0\) to zero at the same time. Therefore the CIR and Vasicek models are not strictly nested. Still, if the Vasicek model cannot be rejected in favour of the general specification (20), it can certainly not be rejected in favour of the CIR model.

The hypothesis \(a_1 = 0\) can be tested in several ways. The easiest method is the Lagrange Multiplier test (see Engle, 1984, for a description and applications), which is computed as \(Tr^2\) of a regression of the squared residuals of the Vasicek model on a constant and the lagged interest rate \(R_\tau(t)\). The critical value of the test is obtained from the \(\chi^2\) table with one degree of freedom.

The system of Eqs. (19) and (20) can be estimated by a conventional feasible Generalized Least Squares estimator, which consists of two rounds of OLS. In the first step Eq. (19) is estimated by OLS. Next the squared residuals are regressed on a constant and the lagged interest rate to obtain first round estimates of \(a_0\) and \(a_1\). The second step produces asymptotically efficient estimates. In the second round the estimated variances from (20) are used to remove the heteroskedasticity in (19) by weighted least squares. Finally the squared residuals of the weighted least squares regression are used to re-estimate Eq. (20) for the final estimates of \(a_0\) and \(a_1\).

2.4. Cross-section estimation

To estimate the parameters on a cross-section of bonds at a given date, we have to deal with the practice that all traded bonds carry coupons. Consider a coupon bond at time \(t\) with price \(P_\tau(c,t)\), which entitles the holder to a vector of \(K\) cashflows \(c = (c_1, \ldots, c_K)\) with corresponding payment dates \(\tau = (\tau_1, \ldots, \tau_K)\). The value of such a bond can be expressed as the value of a portfolio of discount bonds:

\[
P^*_\tau(c, t) = \sum_{j=1}^{K} c_j P_{\tau_j}[r(t)]
\]
where the discount bond prices are given by (6) for the Vasicek model and by (13) for the CIR model. Taken literally, the term structure models imply that all bonds are a deterministic function of the instantaneous spot rate without any error term. A justification of a stochastic error term is the presence of measurement error due to the bid–ask spread, and the existence of transactions costs. To estimate the parameters of the model at time \( t \), we assume like Brown and Dybvig (1986) that the quoted bond price \( P_i(c,t) \) deviates from the model price \( P^*_i(c,t) \) by an error term with zero mean. We assume that the error terms for different bonds are independent and have constant variance. In that case we can estimate (21) by nonlinear least squares (NLS) by minimizing the sum of squared errors

\[
S^2_t = \sum_{i=1}^{n} \left[ P_{i,t}(c,t) - P^*_{i,t}(c,t) \right]^2
\]

over the structural parameters \( \kappa, R_\infty, \) and \( \sigma^2 \) and the risk-free rate \( r(t) \) for the \( n \) different bonds that are actively traded at time \( t \). Since the instantaneous spot rate \( r(t) \) is unobservable, we treat, following Brown and Dybvig (1986), \( r(t) \) as an additional unknown parameter, which is estimated jointly with the structural parameters.

Eq. (21) can be estimated for each trading day. Preliminary estimation revealed that the parameters of both models were hardly estimable using data for a single trading day. We therefore decided to pool the data, keeping the structural parameters constant for several days. Our first pooling interval is one week, in which case the parameters \( \kappa, \sigma^2, \) and \( R_\infty \) are constant for Monday through Friday of a particular week. The risk-free rate is allowed to take on a different value each day. We also considered the longer interval of a month. Again \( \kappa, \sigma^2, \) and \( R_\infty \) are constant for the whole period, while \( r(t) \) is different for each trading day. The total number of parameters that are estimated simultaneously is thus equal to the number of trading days in the period plus 3.

3. Data description

For the cross-section estimation of the models of Vasicek (1977) and Cox et al. (1985) we use data of actively traded Dutch government bonds for each trading day during 1989 and 1990. For each bond we have data on the clean closing price

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8 Alternative stochastic specifications are possible. Homoskedastic errors for prices imply heteroskedastic errors for yields. An error in a bond price has the largest effect on yields for short-term bonds. With our specification, long-term yields fit closer than short-term yields. If we aim to fit yields we would have to assume that the errors in our specification were proportional to squared duration. This considerably affects all estimates. Our assumption of homoskedastic errors in prices is consistent with the interpretation of measurement error due to the bid–ask spread, which is constant across maturities.
Table 1
Summary of bond data

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Std. dev.</th>
<th>Minimum (date)</th>
<th>Maximum (date)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max yield</td>
<td>7.99</td>
<td>0.90</td>
<td>6.40 (89/01/02)</td>
<td>9.30 (90/12/31)</td>
</tr>
<tr>
<td>Min yield</td>
<td>7.65</td>
<td>0.90</td>
<td>5.52 (89/01/05)</td>
<td>8.79 (90/09/28)</td>
</tr>
<tr>
<td>Sd. yield</td>
<td>0.08</td>
<td>0.04</td>
<td>0.03 (89/03/13)</td>
<td>0.19 (89/12/18)</td>
</tr>
<tr>
<td>Spread</td>
<td>−0.14</td>
<td>0.25</td>
<td>−0.95 (89/10/10)</td>
<td>0.67 (89/01/05)</td>
</tr>
<tr>
<td>Maturity</td>
<td>9.97</td>
<td>0.06</td>
<td>9.81 (89/10/25)</td>
<td>10.13 (90/03/22)</td>
</tr>
<tr>
<td>Duration</td>
<td>7.21</td>
<td>0.25</td>
<td>6.81 (89/10/08)</td>
<td>7.68 (89/01/02)</td>
</tr>
<tr>
<td>Number of bonds</td>
<td>40.13</td>
<td>4.12</td>
<td>33 (89/01/02)</td>
<td>47 (90/10/31)</td>
</tr>
</tbody>
</table>

Notes. The table contains summary statistics of the Dutch bond price data. The sample period is 89/01/02 to 90/12/31, a total of 507 daily observations. Max yield, Min yield and Sd. yield refer to the maximum yield, the minimum yield and the standard deviation of yields on a given day. Spread is the difference in yield between the bond with the longest and shortest duration on a day. Maturity and duration give the maximum maturity and duration on a day. In parentheses is the date on which the minimum and the maximum occurred.

and accrued interest. Based on this we computed the cashflow patterns consisting of coupon payments, final repayment and corresponding payment dates.

Table 1 provides summary statistics of the dataset. The total number of trading days is 507, while the number of actively traded bonds on a day varies between 33 and 47. The longest maturity in this dataset is about 10 years, while the duration of the longest bonds is on average somewhat more than seven years. Most of the period the yield curve has been very flat, and on average it had an inverted shape.

For the estimation of the time series models we use daily observations of the one-month Amsterdam InterBank Offered Rate (AIBOR). Fig. 1A shows a graph of this series for the sample period corresponding to our cross-section data. Interest rates were steadily rising during this two-year period. Because of the trend like behaviour over this short period it is impossible to obtain any reliable estimate of the mean reversion parameter and the unconditional mean of the short-term interest rate. Fig. 1B shows the time series for the longer sample period 1985 to 1991.

4. Results

4.1. Time series

In order to find significant mean reversion ($\kappa > 0$) one needs long time series, spanning many years. Since the discrete time representation of the process of the
risk-free rate is a first-order autoregression, a test for $\kappa = 0$ amounts to testing for a unit root. For the Vasicek model we could use the test in Fuller (1976). According to table 8.5.1 in Fuller (1976) the 5% critical value of the test statistic $n(\hat{\vartheta} - 1)$ is $-14.1$, where $n$ is the number of observations, and $\hat{\vartheta}$ the estimated
Table 2  
Results of time series estimation

<table>
<thead>
<tr>
<th>Maturity</th>
<th>1/12</th>
<th>2/12</th>
<th>3/12</th>
<th>6/12</th>
<th>12/12</th>
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</thead>
<tbody>
<tr>
<td>The model of Vasicek</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b_0$</td>
<td>7.50</td>
<td>7.51</td>
<td>7.45</td>
<td>7.40</td>
<td>7.52</td>
</tr>
<tr>
<td></td>
<td>(8.21)</td>
<td>(7.22)</td>
<td>(6.92)</td>
<td>(6.45)</td>
<td>(6.86)</td>
</tr>
<tr>
<td>$b_1$</td>
<td>0.99799</td>
<td>0.99794</td>
<td>0.99778</td>
<td>0.99762</td>
<td>0.99778</td>
</tr>
<tr>
<td></td>
<td>(0.0030)</td>
<td>(0.0029)</td>
<td>(0.0031)</td>
<td>(0.0031)</td>
<td>(0.0029)</td>
</tr>
<tr>
<td>$x^2 \times 10^{-3}$</td>
<td>4.23</td>
<td>3.55</td>
<td>3.85</td>
<td>3.78</td>
<td>3.76</td>
</tr>
<tr>
<td></td>
<td>(0.16)</td>
<td>(0.13)</td>
<td>(0.14)</td>
<td>(0.14)</td>
<td>(0.14)</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.503</td>
<td>0.516</td>
<td>0.557</td>
<td>0.596</td>
<td>0.556</td>
</tr>
<tr>
<td></td>
<td>(0.297)</td>
<td>(0.268)</td>
<td>(0.278)</td>
<td>(0.278)</td>
<td>(0.264)</td>
</tr>
<tr>
<td>$\sigma^2$</td>
<td>1.104</td>
<td>0.967</td>
<td>1.108</td>
<td>1.266</td>
<td>1.601</td>
</tr>
<tr>
<td></td>
<td>(0.049)</td>
<td>(0.057)</td>
<td>(0.089)</td>
<td>(0.186)</td>
<td>(0.467)</td>
</tr>
</tbody>
</table>

The model of Cox, Ingersoll and Ross

| $b_0$    | 7.38 | 7.34 | 7.27 | 7.17 | 7.27 |
|          | (7.78) | (6.80) | (6.48) | (5.74) | (6.09) |
| $b_1$    | 0.99789 | 0.99782 | 0.99764 | 0.99743 | 0.99755 |
|          | (0.00120) | (0.00109) | (0.00113) | (0.00110) | (0.00110) |
| $a_0$    | 2.313 | 1.470 | 1.242 | 0.304 | 0.340 |
|          | (1.457) | (1.125) | (1.277) | (1.256) | (1.103) |
| $a_1$    | 3.036 | 3.278 | 4.105 | 5.414 | 5.271 |
|          | (2.254) | (1.718) | (1.958) | (1.908) | (1.656) |
| $x^2 \times 10^{-3}$ | 4.55 | 3.87 | 4.23 | 4.18 | 4.17 |
|          | (0.50) | (0.42) | (0.48) | (0.51) | (0.49) |
| LM(1)    | 0.82 | 1.50 | 2.32 | 5.18* | 6.67* |
| $\kappa$ | 0.527 | 0.545 | 0.591 | 0.644 | 0.613 |
|          | (0.301) | (0.273) | (0.284) | (0.276) | (0.275) |
| $\sigma^2$ | 0.155 | 0.132 | 0.146 | 0.146 | 0.144 |

Notes. The table contains the estimated coefficients and structural parameters of the Vasicek and CIR models. The sample period is 85/01/02 until 90/12/31, which contains 1551 daily observations of AIBOR rates. The maturity is expressed in years; the estimated coefficients and structural parameters are measured in percent per annum. Standard errors are given in parentheses. LM(1) denotes the value of the test statistic of the Lagrange Multiplier test, which is asymptotically distributed as $\chi^2$ with one degree of freedom. An asterisk denotes significance at the 5% level.

first-order autocorrelation. Substituting $\kappa = -\ln(\rho)/\Delta t$, using $\ln(1 + x) = x$, and noting that $n\Delta t$ is the number of years (six for our sample), we find the inequality $\kappa > 2.35$, which is implausibly large. For smaller values of $\kappa$ the unit root hypothesis, implying the Merton (1973) model for the term structure, can not be rejected.

The 1550 observations in the six year sample are more informative about the volatility of interest rates. The results are in Table 2. The asymptotic standard error on $s^2 = \text{var}(\eta_t(t, t + \Delta t))$ in the Vasicek model is of the order $\sigma^2 \sqrt{2}/T$, which is 3.5% of the estimated value $s^2$. For the CIR model the unconditional variance $s^2 = a_0 + a_1 b_0$ is of the same order of magnitude. The Lagrange
Multiplier test rejects the constant volatility Vasicek model in favour of the heteroskedasticity implied by the CIR model for the two longest maturities. Rejection of the Vasicek model does of course not necessarily indicate that the CIR model must be accepted. It merely tells us that there is some heteroskedasticity. Alternative specifications like ARCH disturbances or the whole range of variance functions estimated in Chan et al. (1992) have not been investigated here. Surprisingly our rejection of the Vasicek model is not so strong as Chan et al. (1991, 1992) have found for the U.S. one-month T-bill rate and the Japanese three-month Gensaki rate.

The last two rows in both panels of Table 2 present estimates of the identifiable structural parameters. The mean reversion parameter is approximately the same for all maturities as it should be. The point estimates are not sensitive to the heteroskedasticity correction applied in estimating the CIR model, but the standard errors become much smaller.

The implied variance $\sigma^2$ of the Vasicek model is estimated very precisely for all individual maturities. The estimates, however, increase with maturities. The variance of the 12 months rate is about 1.5 times the variance of the one month rate. The increasing estimates of the instantaneous variance are the same as the excess volatility phenomenon first noted by Shiller (1979), and which started an enormous literature. Excess volatility of long-term interest rates means that the actual term structure of volatilities is much less decreasing than is implied by the term structure of volatilities defined below Eq. (18). Long-term rates fluctuate much more than is implied by the volatility of the shortest rate.

For the CIR model it proved numerically impossible to solve the set of nonlinear equations for the structural parameters, given estimates of $a_0, a_1, b_0,$ and $b_1$, as one of the equations in the system is nearly redundant. The estimates in Table 3 are obtained by approximating $\theta$ (the unconditional mean of the risk-free rate) by $b_{0 \tau}$ (the unconditional mean of the yield of a discount bond with maturity $\tau$). The approximation error is likely to be very small given that the term structure was almost flat over most of the sample period. The CIR estimates of $\sigma^2$ are much more constant across maturities and do not exhibit the excess volatility phenomenon. In this respect the CIR model performs better than the Vasicek model. Time series analysis of the daily interest rates is not the principal aim of this paper though. We have estimated the time series models primarily to obtain some benchmark parameter values to compare with the results of the cross-section estimates to be discussed below. An AR(1) specification appears adequate for the

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9 The estimates of $\sigma^2$ are also sensitive to the method of aggregation of the continuous time process. The entries in the table are computed by exact temporal aggregation as described in Section 2.3. The approximate estimates, obtained by simply dividing the residual standard error by the time between successive observations, are smaller by between 5% for the one month rate and 40% for the 12 months rate.

10 See LeRoy (1989) for an overview.
Table 3  
Summary of weekly pooled cross-sectional estimation

<table>
<thead>
<tr>
<th></th>
<th>( r(t) )</th>
<th>( \kappa )</th>
<th>( R_\infty )</th>
<th>( \sigma^2 )</th>
<th>( \sigma )</th>
<th>s.e.</th>
</tr>
</thead>
<tbody>
<tr>
<td>The model of Vasicek</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>7.72</td>
<td>0.918</td>
<td>6.11</td>
<td>1763.2</td>
<td>18.96</td>
<td>0.1651</td>
</tr>
<tr>
<td>Std. dev.</td>
<td>1.36</td>
<td>1.224</td>
<td>3.20</td>
<td>4722.0</td>
<td>37.47</td>
<td>0.0397</td>
</tr>
<tr>
<td>Minimum</td>
<td>3.15</td>
<td>0.010</td>
<td>0.50</td>
<td>0.05</td>
<td>0.22</td>
<td>0.0884</td>
</tr>
<tr>
<td>Maximum</td>
<td>11.24</td>
<td>5.000</td>
<td>17.15</td>
<td>35104.6</td>
<td>187.36</td>
<td>0.2965</td>
</tr>
<tr>
<td>Median</td>
<td>7.76</td>
<td>0.187</td>
<td>6.94</td>
<td>0.05</td>
<td>0.22</td>
<td>0.1623</td>
</tr>
<tr>
<td>Range</td>
<td>2.03</td>
<td>1.832</td>
<td>2.74</td>
<td>2.73</td>
<td>14.22</td>
<td>0.0521</td>
</tr>
<tr>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>The model of Cox, Ingersoll and Ross</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>7.88</td>
<td>0.430</td>
<td>6.85</td>
<td>0.866</td>
<td>11.71</td>
<td>0.1653</td>
</tr>
<tr>
<td>Std. dev.</td>
<td>1.11</td>
<td>0.811</td>
<td>2.47</td>
<td>2.086</td>
<td>22.29</td>
<td>0.0398</td>
</tr>
<tr>
<td>Minimum</td>
<td>5.61</td>
<td>0.010</td>
<td>0.40</td>
<td>0.0001</td>
<td>0.076</td>
<td>0.0883</td>
</tr>
<tr>
<td>Maximum</td>
<td>10.40</td>
<td>5.000</td>
<td>18.57</td>
<td>12.484</td>
<td>95.19</td>
<td>0.3053</td>
</tr>
<tr>
<td>Median</td>
<td>8.00</td>
<td>0.085</td>
<td>6.95</td>
<td>0.0001</td>
<td>0.300</td>
<td>0.1638</td>
</tr>
<tr>
<td>Range</td>
<td>2.14</td>
<td>0.557</td>
<td>3.78</td>
<td>0.003</td>
<td>1.202</td>
<td>0.0659</td>
</tr>
</tbody>
</table>

Notes. Sample period is 89/01/02 to 90/12/31, total of 104 weekly cross-sections. Minimum and maximum refer to the minimum and maximum value over the 104 weeks. Median and range refer to the median and interquartile range of the 104 estimated values. s.e. refers to the pricing error in guilders. \( r(t) \) refers to the weekly mean of the riskfree rate over a given month. Units for the risk-free rate \( r(t) \), the long-term yield \( R_\infty \), and the volatility \( \sigma \) (\( \sigma\sqrt{r(t)} \) for CIR) are percent per annum.

dynamics of the short-term interest rates, since for all five series the second-order partial autocorrelation coefficient proves insignificant.

4.2. Weekly pooled cross-sections

The results of the 104 weekly estimates are summarized in Table 3 and Figs. 2 and 3. Both models, CIR and Vasicek, provide a good fit for bond prices. The average pricing error is about 0.17 guilders (par bonds are normalized to 100 guilders). As Fig. 3D demonstrates, the fit of the two models is almost equal for every week in the sample. The outlier in the figure is the last week of the sample, which contains the last three trading days of December 1990. Apart from this week the two models are statistically indistinguishable. The fit is also fairly constant over the weeks; the minimum of 9 cents is not far from the bid–ask spread; the maximum standard deviation of the pricing errors is only 30 cents.

The scatter diagrams in Figs. 3A–3C show some extreme outliers in the parameter estimates and many points on the boundary of the admissible parameter space. Because of the frequent occurrence of outliers the mean and standard deviation of the parameter estimates over the 104 weeks are not very informative. The median and interquartile range are more robust measures of location and dispersion.
Fig. 2 shows the estimates of the risk-free rate. Again the two models are very similar, although the Vasicek model produces some outliers. The estimated risk-free rate is close to the observed one month AIBOR rate. Standard errors on each $r(t)$, not reported here, are generally small. ¹¹

At the other end of the yield curve the infinite maturity yield $R_\infty$ can also be estimated reasonably well. The scatter diagram of Fig. 3B shows the similarity of the Vasicek and the CIR models. Both models produce the same three upward outliers; these are all in the first month of the sample. In addition the Vasicek model is plagued by a series of downward outliers, which are dated mostly in the last quarter of 1990. These outliers all occur when $\kappa$ is estimated to be very small. If left unconstrained the estimates of $R_\infty$ then all converge to very large negative numbers. We therefore set a numerical lower bound of 0.5%. In Fig. 3B there are

¹¹ Standard errors of the parameters have been computed using the inverse of outer product of the scores. The standard error of $R_\infty$ is either of the same order as those on $r(t)$ or extremely large (when $\kappa$ is small). The other parameters are never significantly different from zero. Still the standard errors are very unreliable and probably an underestimate of the true standard errors for two reasons. Due to the pooling of the five days of the week it is likely that the errors are autocorrelated. Second, for most weeks we obtain a corner solution where some of the parameters are constrained. In that case we can only obtain standard errors conditional on the constrained parameters, and these are an order of magnitude smaller than the unconditional standard errors that we obtain in case of an interior optimum.
20 points cluttered along the minimum value of the Vasicek axis. For the CIR model a negative estimate of $R_\infty$ occurs only once; it too was constrained to 0.5%. Because of this, the mean of $R_\infty$ in Table 3 is smaller for the Vasicek model than for the CIR model; the median values are the same though. Apart from the outliers the implication that the long-term yield should be constant over time holds quite well. The variability of the estimates could be nothing more than sampling variability. However, the level of $R_\infty$ slowly follows the rise in the short-term interest rate. One interpretation of this result is that the longest bond in our sample only has a maturity of 10 years, and this can be too small to be representative of a very long-term bond, especially if the mean reversion parameter $\kappa$ is small.

Note that with $\kappa = 0.01$, which is the estimate of $\kappa$ in about 25% of the weekly samples, the weight $w_1(\tau)$ of the short rate in a 10 year bond ($\tau = 10$) is still equal
to 0.95, so that the long-term yield $R_\infty$ with a weight of $1 - w_1 = 0.05$ does not contribute much to the overall yield. Hence $R_\infty$ can not be estimated with any precision if $\kappa$ is small. Although for the CIR model the weights of the risk-free rate and the long-term yield are analytically more complicated—see Eq. (15)—they turn out to be very similar. With $\tilde{\kappa} = 0.01$ and $\sigma^2 = 10^{-4}$, we find $v_1(\tau = 10) = 0.95$ and $v_3(\tau = 10) = 0.06$.

The mean reversion parameter $\kappa$ (or $\tilde{\kappa} = \kappa - \lambda$ in the CIR model) and the volatility parameter $\sigma^2$ behave erratically. They are usually very different from week to week, and have large standard errors. Since the term structure models are only valid for $\kappa > 0$, and since $R_\infty$ becomes unidentified as $\kappa \to 0$, and since it is numerically impossible to compute bond prices for very small $\kappa$, we had to restrict $\kappa$ (and similarly $\tilde{\kappa}$) to $\kappa > 0.01$. This corner solution occurs for both models in about 45% of all weeks. If we find a small $\kappa$ in the Vasicek model, we usually also hit the lower bound for $R_\infty$. Such a behaviour for $R_\infty$ is consistent with the Merton (1973) model. In that model the spot rate is assumed to follow a standard Brownian motion without any mean reversion. One of the implications of the Merton model is that the infinite maturity yield $R_\infty$ goes to minus infinity. On the other hand, large values of $\kappa$ are also unacceptable. If $\kappa > 5$ the Hessian of the likelihood function becomes numerically singular, since with such a strong mean reversion the spot rate is almost white noise and the risk-free rate $r(t)$ drops out of the model, since both $v_1(\tau)$ (CIR model) as well as $w_1(\tau)$ (Vasicek) go to zero for all $\tau$. The upper bound is attained three times. When the optimum of $\kappa$ falls in the admissible range, the estimates of $\tilde{\kappa}$ are smaller or equal to the corresponding estimates of $\kappa$ in the Vasicek model (see Fig. 3A). The inequality $\tilde{\kappa} < \kappa$ is consistent with the implied term premium that is implicit in the CIR model.

In many cases $\sigma^2$ falls to zero in unconstrained optimization. For numerical stability we set a lower bound $\sigma^2 > 10^{-4}$ for the CIR model. For the Vasicek model we set the corresponding bound $\sigma^2 > 5 \times 10^{-6}$. The lower bounds are binding in more than half of the sample weeks. We checked the effect of the lower bound on the fit of the models, which turned out to be negligible. The standard deviation of the price residuals never changes by more than one-tenth of a cent if the bound would have been set at $10^{-3}$ or even $10^{-2}$, indicating that the likelihood function is exceptionally flat. Brown and Dybvig (1986) and Brown and Schaefer (1994) encountered similar problems. For those weeks where we find an unconstrained optimum the CIR and Vasicek models are again very much alike.

The overall impression from the cross-section estimates is that the model is overspecified. Without any loss in fit we can set one of the structural parameters at some "reasonable" value and optimize over the others. The symptoms of the over-parameterization are the frequent occurrence of outliers and the near singularity of the Hessian of the least squares fitting function. In the next subsection we will investigate whether we can reduce the overfitting problem by extending the length of the pooling period from one week to one month, which constrains the
Table 4
Summary of monthly pooled cross-sectional estimation

\[ r(t) \quad \kappa \quad R_x \quad \sigma^2 \quad \sigma \quad \text{s.e.} \]

<p>| | | | | | | |</p>
<table>
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<tbody>
<tr>
<td>The model of Vasicek</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>8.18</td>
<td>0.116</td>
<td>21.94</td>
<td>14.87</td>
<td>2.84</td>
<td>0.1681</td>
</tr>
<tr>
<td>Std. dev.</td>
<td>0.88</td>
<td>0.128</td>
<td>26.75</td>
<td>26.01</td>
<td>2.61</td>
<td>0.0310</td>
</tr>
<tr>
<td>Minimum</td>
<td>6.42</td>
<td>0.010</td>
<td>6.96</td>
<td>0.05</td>
<td>0.22</td>
<td>0.1223</td>
</tr>
<tr>
<td>Maximum</td>
<td>9.21</td>
<td>0.490</td>
<td>107.23</td>
<td>105.57</td>
<td>10.27</td>
<td>0.2300</td>
</tr>
<tr>
<td>Median</td>
<td>8.58</td>
<td>0.070</td>
<td>7.25</td>
<td>4.74</td>
<td>2.17</td>
<td>0.1607</td>
</tr>
<tr>
<td>Range</td>
<td>1.60</td>
<td>0.155</td>
<td>20.35</td>
<td>11.46</td>
<td>2.45</td>
<td>0.0406</td>
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</table>

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<tbody>
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<td>The model of Cox, Ingersoll and Ross</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>7.93</td>
<td>0.190</td>
<td>7.71</td>
<td>0.180</td>
<td>3.56</td>
<td>0.1784</td>
</tr>
<tr>
<td>Std. dev.</td>
<td>0.96</td>
<td>0.319</td>
<td>3.91</td>
<td>0.599</td>
<td>11.19</td>
<td>0.0360</td>
</tr>
<tr>
<td>Minimum</td>
<td>5.80</td>
<td>0.010</td>
<td>1.78</td>
<td>0.0001</td>
<td>0.05</td>
<td>0.1195</td>
</tr>
<tr>
<td>Maximum</td>
<td>9.01</td>
<td>1.255</td>
<td>20.68</td>
<td>2.306</td>
<td>42.71</td>
<td>0.2390</td>
</tr>
<tr>
<td>Median</td>
<td>8.34</td>
<td>0.033</td>
<td>6.89</td>
<td>0.0001</td>
<td>0.07</td>
<td>0.1759</td>
</tr>
<tr>
<td>Range</td>
<td>1.82</td>
<td>0.147</td>
<td>1.83</td>
<td>0</td>
<td>0.01</td>
<td>0.0538</td>
</tr>
</tbody>
</table>

Notes. Sample period is 89/01/02 to 90/12/31, total of 24 monthly cross-sections. Minimum and maximum refer to the minimum and maximum value over the 24 months. Median and range refer to the median and interquartile range of the 24 estimated values. S.e. refers to the pricing error in guilders. \( r(t) \) refers to the monthly mean of the risk-free rate over a given month. Units for the risk-free rate \( r(t) \), the long-term yield \( R_x \), and the volatility \( \sigma \) (\( \sigma r(t) \) for CIR) are percent per annum.

Parameters to be constant over a longer period. In Section 4.5 we will then provide a more precise meaning of "reasonable" value by conducting a grid search.

A second general conclusion is that the typical estimates imply that the implied process of the risk-free rate is close to a random walk without drift and almost deterministic. This conclusion might be specific for our dataset containing many days with flat term structures. Still the low implied volatility conflicts strongly with the time series estimates. Interest rates have been quite volatile during the sample, but the yield curve has shifted up and down horizontally. If the spot rate would be mean reverting the long-term yields must be much less volatile than the risk-free rate, and we should have seen more weeks with a steep downward sloping yield curve, reflecting the historically high current interest rates and the high time series volatility and mean reversion.

4.3. Monthly pooled cross-sections

Because the estimates of the crucial parameters \( \kappa \) and \( \sigma^2 \) are often very poor in the weekly pooled samples, and in order to investigate the possibility of overfitting, we re-estimated the two models by pooling the data for a full month.

The 24 monthly estimates are summarized in Table 4 and Figs. 4 and 5, which are analogous to the presentation of the weekly results. The monthly estimates
differ in many respects from the weekly estimates. The fit is still very good, despite the additional restrictions due to keeping the structural parameters constant over a longer period. For the Vasicek model the average standard deviation of the pricing errors is only 0.3 cents larger than for the weekly pooled estimates; for the CIR model the 1.3 cents deterioration in the fit is somewhat larger. A formal test for constant parameters is not possible, since the weekly samples are not exactly nested within the months; a month is never exactly four weeks.

The key to understanding the differences between the weekly and monthly estimates is the behaviour of the mean reversion parameter \( \kappa \) (and \( \bar{\kappa} \)). The mean reversion parameter is lower than for the weekly estimates, both with respect to the average, the median, as well as the maximum. Whereas in about one third of the weekly samples we find \( \kappa > 1 \) for the Vasicek model, such large values never occur for the monthly samples.\(^{12}\) For the CIR model only one very large estimate of \( \bar{\kappa} \) remains, and several more moderate outliers (see Fig. 5A). Note the very different scale of Fig. 5A as compared to the weekly estimates in Fig. 3A.

\(^{12}\) Recall from the definition of \( \kappa \) in Section 2 that \( \kappa = 1 \) implies a first-order autocorrelation coefficient of \( \rho = 0.37 \) in annual short-term interest rate data. With \( \kappa = 2 \), the autocorrelation coefficient drops to \( \rho = 0.14 \). Both are exceptionally low compared to the time series estimate, which is of the order \( \rho = 0.58 \) (\( \kappa = 0.55 \)).
Fig. 5. Scatter diagrams of the monthly estimated structural parameters of the Vasicek and CIR models on a daily basis for the years 1989 and 1990. The values for the Vasicek model are shown on the x-axis, while the corresponding values for the CIR model are shown on the y-axis.

Consistent with the existence of a risk premium, it still holds that the median CIR estimate $\bar{\kappa}$ is still lower than the median of $\kappa$ in the Vasicek model. All this reinforces the impression that the implied nominal risk-free rate is very close to a random walk. It looks as though the current spot rate is the market's forecast of the future interest rate.

One consequence of the low estimates of $\kappa$ is that the estimates of the long-term yield $R_\infty$ have become more erratic (see Fig. 5B). The outliers always occur when $\kappa$ is very close to zero, in which case $R_\infty$ is not identified. The estimates of $r(t)$ are much more stable, and the large outliers have completely disappeared. Otherwise the risk-free rate is still very much the same as for the weekly estimates, and also very close to the one-month AIBOR rate.
The point estimate of $\sigma^2$ less often hits the lower bound for the Vasicek model. In Fig. 5C the estimates are scattered along the horizontal axis, meaning that the CIR volatility is almost zero, while $\sigma^2$ for the Vasicek takes on values that are of the same order of magnitude as the time series estimates. Again this is due to the lower estimates of $\kappa$; with a small $\kappa$ volatility matters for the shape of the term structure. The scale of the CIR axis in Fig. 5C is very different from the Vasicek scale in order to plot the two outliers in the CIR volatility estimates.

From the results it is hard to prefer one term structure models over the other. If the parameters are revised every week or every month, both models perform equally well. Since the cross-sectional fit is so good, more general models with more than one factor will lead to greater estimation difficulties. The simple one-factor models already fit as well as one might hope, given the bid–ask spread and transaction costs. Single-factor models already appear to be too flexible. The additional parameters in models with more factors introduce a serious danger of fitting noise.

4.4. Option pricing

One of the major applications of the Vasicek and CIR model is the valuation of options on bonds. To compare both models with respect to option valuation, we computed a European at-the-money call option with a maturity of four years on a discount bond with a maturity of eight years. The reason for taking this particular option is to cover the relevant maturity spectrum and to concentrate primarily on the option’s time value.

The value of an interest rate contingent claim is equal to the discounted expected value in a risk-neutral economy of the payout of the claim at maturity. Closed-form solutions are available for the Vasicek and CIR models. Fig. 6 shows the computed option prices, based on the weekly pooled cross-sections. On average, the Vasicek option values are higher than the CIR option values, which relates to the corresponding relationships between the mean reversion and volatility of both models. Fig. 7 shows the computed option prices in case of the monthly pooled cross-sections. Contrary to the weekly pooled cross-sections, Vasicek option values are much higher than the corresponding CIR option values. Because the volatility of the CIR model is very small compared to the Vasicek model, as shown in Fig. 5, this effect is easily understood.

It is interesting to compare the weekly and monthly pooled cross-section option values with the option values based on the time series estimates. However, because it is not possible in case of the Vasicek model to compute option prices

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13 Further lengthening of the pooling interval turned out to be technically infeasible. The longer the sample period, the larger the number of parameters. Every day adds one more risk-free rate. The joint nonlinear estimation of all parameters proved to present too many problems given our current hardware and software algorithms. Further pooling remains a topic of further research.

14 Computational details are available from the authors on request.
Fig. 6. European at-the-money call option values on a weekly basis for the years 1989 and 1990, or 100 option values, based on the weekly cross-sectional estimation. The maturity of the underlying bond is eight years, while the maturity of the option is four years. The values of the time-series option values in case of the Vasicek and CIR model are 0.40 and 0.38, respectively.

using the estimated time-series parameters only, we use the relation between the average one- and two-month rate to identify the extra parameter $\theta$ necessary. In addition, these averages are used to determine for both models the instantaneous short-term rate of interest. The resulting Vasicek and CIR option values, as shown in Figs. 6 and 7, are 0.40 and 0.38, respectively. On average, these option values are lower than their cross-sectional counterparts, a result which can be explained by the relatively high value of the mean reversion parameter obtained in the time series analysis which dominates the higher time series volatility.

4.5. A closer look at a selected week

The critical parameter causing the numerical estimation problems is $\kappa$ (or $\bar{\kappa}$). Conditional on $\kappa$ both the Vasicek model as well as the CIR model can be easily estimated.\textsuperscript{15} Another way to learn more about the properties of the model is to

\textsuperscript{15} The problem remains nonlinear, because we fit prices instead of yields, and because we have to deal with coupon paying bonds. But conditionally on $\kappa$, the likelihood function is very close to a quadratic function, and optimization poses no numerical problems.
investigate how the model's implications and fit are affected by setting $\kappa$ (or $\tilde{\kappa}$) at some prespecified values. For this purpose we selected the week October 22–26, 1990 (week 9043), which week is in many respects typical for the Dutch term structure in the two-year sample period. For that week the yield curve is flat, as shown in Fig. 8A. The parameters are very poorly estimated for both models with $\kappa$ and $R_\infty$ hitting a lower bound. Fig. 8B shows the pricing residuals, which do not show any particular pattern. The actual and fitted prices in Fig. 8C show that the models fit perfectly in the price dimension with $R^2 = 0.999$ for both models.\footnote{The fitted values in the figures are for the CIR model. The Vasicek results are not shown, because they are not noticeably different.} The fit is very poor, however, in the yield dimension with an $R^2 = 0.030$, implying that the models can explain only 3% of the cross-sectional variation in yields.

Fig. 9 shows the results of a grid search over $\kappa$. Conditional on a range of values for $\kappa$ we estimated the other parameters of the model. Fig. 9A shows the fit of the models, measured by the root mean squared error (RMSE) of the price residuals. The CIR and Vasicek models are almost indistinguishable for most
Fig. 8. Some results of the estimation in case of the particular week October 22–26, 1990, which is in many respects typical for the Dutch bond market.
Fig. 9. Some results of the grid search over the mean reversion parameter under the Vasicek and Cox et al. models for the particular week October 22–26, 1990, which is in many respects typical for the Dutch bond market.
values of $\kappa$ (and $\bar{\kappa}$). The higher sum of squared errors for the Vasicek model with very small values of $\kappa$ is due to the binding constraint on $R_\infty$: for $\kappa = 0.03$ the unconstrained estimate of $R_\infty$ is negative. The global minimum is obtained when $\kappa \to 0$. The most disturbing feature is the decrease of the function value for large $\kappa$. This explains why it is so hard to obtain point estimates, and also why we sometimes find several local optima. If the starting value of $\kappa$ in the optimization search in larger than 0.3, the algorithm will not converge, and produce implausibly large estimates of $\kappa$. The shape of the least squares function is similar in other weeks, and also explains why we sometimes find the large outliers for $\kappa$ and $\bar{\kappa}$.

The other structural parameters are also identical for both models. Fig. 9B shows the implied volatility. Week 9043 is special in one respect, because there is a range of values for which the volatility estimate is not at its lower bound. Finally, the bottom graph of Fig. 9 shows the call option values for the two models for all values of the mean reversion parameter. The maturity of the at-the-money option and the underlying discount bond is again four and eight years, respectively. Although a small part of the difference between the option values may be explained by the binding constraint on $R_\infty$ in case of the Vasicek model for low values of the mean reversion parameter, the overall significant difference is striking given the similarity between the volatility parameters, and the similarity found in the cross-section estimates.

5. Conclusions

In this paper we empirically compared the models of Vasicek (1977) and Cox et al. (1985). We estimated the term structure of interest rates implied by both models using a cross-section of liquid Dutch government bonds of 1989 and 1990.

For the weekly cross-section estimates the two models are almost indistinguishable: they fit equally well, and lead to the similar implied option values. In general the parameters are numerically difficult to estimate, especially for the CIR model. It seems that the simple one-factor models are already overspecified. Restricting the models further, by pooling over a full month, showed more differences between the models. The Vasicek model now fits slightly better. The main difference between the two models is in the volatility estimates, which are extremely low for the CIR model, and which show up clearly in the implied option values.

Comparing the cross-sectional estimation results with time series results, we find that the mean reversion parameter is small in both sets of results, and hardly distinguishable from zero (the random walk case). On the other hand, volatility is much more precisely estimated from high-frequency time series data than as an implied volatility from bond prices.

An interesting topic for future research is the empirical comparison between term structure models based on an exogenous term structure of interest rates like
Ho and Lee (1986) and Heath et al. (1992). The estimated volatility structure of interest rates and the resulting option values can then be compared with models estimated in this paper. A second topic for future research is the efficient combination of the time series and cross-sectional data information using a formal panel data setup. This problem is studied in Schotman (1994).

References


Schotman, P.C., 1994, A Bayesian approach to the empirical valuation of bond options, LIFE working paper, University of Limburg.