Optimal bundle pricing with monotonicity constraint

A. Grigoriev\textsuperscript{a}, J. van Loon\textsuperscript{a}, M. Sviridenko\textsuperscript{b}, M. Uetz\textsuperscript{c,*}, T. Vredeveld\textsuperscript{a}

\textsuperscript{a} Maastricht University, Quantitative Economics, P.O. Box 616, 6200 MD Maastricht, The Netherlands
\textsuperscript{b} IBM T.J. Watson Research Center, P.O. Box 218, Yorktown Heights, NY 10598, USA
\textsuperscript{c} University of Twente, Applied Mathematics, P.O. Box 217, 7500 AE Enschede, The Netherlands

\textbf{A R T I C L E  I N F O}

\textbf{Article history:}
Received 8 November 2007
Accepted 23 April 2008
Available online 19 June 2008

\textbf{Keywords:}
Bundle pricing
Revenue optimization
Monotonicity
Computational complexity
Approximation algorithm

\textbf{A B S T R A C T}

We consider the problem of pricing (digital) items in order to maximize the revenue obtainable from a set of bidders. We suggest a natural monotonicity constraint on bundle prices, show that the problem remains NP-hard, and we derive a PTAS. We also briefly discuss the highway pricing problem.

© 2008 Elsevier B.V. All rights reserved.

1. Introduction

We consider the situation where we want to sell a set of (digital) items to a set of bidders. Every bidder places bids on subsets, or bundles of items, and each bidder would like to receive one or more of these bundles (OR-bids). Bidders have valuations for each of the bundles they bid on. The valuation is the maximum amount a bidder is willing to pay for a particular bundle. We assume that the bundles and the valuations are known. Hence we are faced with a purely algorithmic problem, in contrast to mechanism design problems where the valuations are private information to the bidders. We assume that each item is available in unlimited supply, as for example in digital goods. We address a standard pricing regime often addressed in the literature, namely each item needs to receive an anonymous price, and the price of any bundle is the sum of its item prices. The optimization problem is to determine single item prices so as to maximize the total revenue collected from all bidders. Notice that the problem is nontrivial only because not all bids need to be granted in an optimal solution, as otherwise the problem reduces to solving a linear program. Also notice that any solution will be trivially envy-free, meaning that a bidder not receiving a particular bundle cannot afford it. This is an artifact of the unlimited supply of items.

In a sequence of recent papers [1,2,5,7,10,13,14], several algorithms and complexity results have been derived for such price optimization problems. Many of the positive results achieve only logarithmic approximation guarantees, and often somewhat trivial algorithms suffice to achieve such bounds, e.g. uniform prices for all items. Demaine et al. [6] delivered a reason for this observation on the performance of algorithms, showing that – in general – the problem does not allow for semi-logarithmic approximation algorithms. In this paper, we introduce a very natural monotonicity constraint on bundle prices that allows us to derive results that break this semi-logarithmic inapproximability barrier. More specifically, we impose the condition that the price of any bundle of size \( k \) must not exceed the price of a bundle of size \( k + 1 \) or larger, for any \( k \). As we will show, this condition implies that most of the items are comparable in the sense that their prices cannot differ too much. Before we elaborate on related work and our contribution in more detail, let us define the pricing settings more formally.

1.1. Model

Let \( I = \{1, \ldots, m\} \) denote the set of items for sale, and let \( J = \{1, \ldots, n\} \) denote the set of bids placed by all bidders. Each bid \( j \in J \) is on exactly one subset of items \( I_j \subseteq I \). In line with notation in auction literature, we call the set \( I_j \) also a bundle. Every bidder has a positive valuation for each of her bundles, that is, every bundle \( I_j \) corresponding to bid \( j \in J \) has a positive valuation \( b_j \) which is the maximum amount its bidder is willing to pay for bundle \( I_j \). We may assume without loss of generality that \( b_j \geq 1 \) for \( j \in J \). By \( p_i \) we denote the price of item \( i \), for \( i = 1, \ldots, m \). The price of a bundle...
Given are \( n \) bids of the form \((i, b_j)\), \( j \in J \). Each bid \( j \) has valuation \( b_j \) and is placed on a bundle \( I_j \subseteq I \), where \( I \) is a set of \( m \) items of unlimited availability each. A solution consists of a price \( p_i \) for every item \( i \in I \). Given item prices, let \( W \subseteq J \) denote all winning bids, so \( \sum_{i \in W} p_i \leq b_j \) for all \( j \in W \) and \( \sum_{i \in W} p_i > b_j \) for all \( j \in J \setminus W \). The objective is to maximize the total revenue \( \sum_{i \in W} \sum_{j \in I} p_i \).

This problem is in general not approximable to within a (semi-) logarithmic factor in the number of bids \( n \) [6]. We therefore introduce a monotonicity constraint on the set of feasible price vectors. More specifically, we impose the following condition for any two subsets of items \( I' \) and \( I'' \).

\[
\sum_{i \in I'} p_i \leq \sum_{i \in I''} p_i \quad \text{whenever } |I'| < |I''|. \tag{1}
\]

The condition has a meaningful economic interpretation in settings where items are different, yet comparable. It only requires that (most) item prices are of the same order of magnitude; see Lemma 6. We therefore suggest in [6] a PTAS with a time complexity of \( O(nm^{3/2} \log B^{3/2}) \), where \( B = \max_b b_j \).

### 2. Arbitrary bundles

We settle the computational complexity of the pricing problem with monotonicity constraint by first showing strong NP-hardness, and then deriving a PTAS.

#### 2.1. Complexity

We show that the pricing problem with monotonicity constraint is strongly NP-hard by using a reduction from the strongly NP-hard problem IndependentSet [8]. Let \( G = (V, E) \) be a graph in which we want to find a maximum cardinality set of vertices that are pairwise non-adjacent. We define an instance \( J \) of the pricing problem as follows. Let \( M \) be an integer that is large enough. For every vertex \( v \in V \) we create a vertex-item, and for every edge \( e \in E \) we introduce an edge-item, that is, \( I = V \cup E \). For every item \( i \in I \), there are \( M + 2 \) bids placed on the bundle consisting of only this item. One of these bids has valuation \( M \), and the others have the same valuation \( M + 1 \). Moreover, for every edge \( e = (u, v) \in E \), there are four more bids. One bid is on bundle \([u, e] \), one bid on bundle \([v, e] \), and two bids are on bundle \([u, v] \). These four bids each have valuation \( 2M + 1 \) each. Here, \( M \) is an integer large enough, for example, \( M \geq 2|E|^2 + 4|E| + 2 \) suffices.

In the NP-hardness proof (Theorem 5) we need the property that in an optimal solution to the above created instance \( J \) of the pricing problem all prices are either \( M \) or \( M + 1 \). Therefore, we first collect some preliminary observations.

#### Lemma 2. In the optimal solution to the pricing problem on \( J \), all prices are within the interval \([M, M + 1]\).

**Proof.** Assume the claim is not true, and let \( p \) be an optimal solution that violates it. If all items have prices \( p_i \leq M \) and there exists at least one item with price \( p_i < M \), then it can easily be verified that the price vector \( p' \), defined by \( p'_i = \max(p_i, M) \), yields the same set of winning bids and the revenue has increased compared to \( p \). Hence, \( p \) cannot be optimal.

Therefore, assume that at least one item, \( h \), has a price \( p_h > M + 1 \) and consider the price vector \( p' \) defined by \( p'_i = p_i \) for \( i \neq h \) and \( p'_h = M + 1 \). As item \( h \) belongs to at most \( 3|E| \) bundles requesting two items, the decrease in revenue of the two item-bundles due to the price change is at most \( 3|E|M \). On the other hand, \( M + 1 \) bidders can now afford their bundle requesting item \( h \). Therefore, the increase in revenue due to these bidders is \( M^2 + 2M + 1 \). Thus, the total revenue increases by at least \( M^2 + 2(3 - 3|M|)M + 1 \) which is positive since \( M > 2|E|^2 \). Hence, the price vector \( p \) cannot be optimal. \( \square \)

#### Lemma 3. In an optimal solution to the pricing problem on \( J \), all bids on two-item bundles are winning bids.
**Lemma 3.** In an optimal solution to the pricing problem on $I$, all prices are integral.

**Proof.** By Lemma 2 we know that each item has a price $p_i = M + d_i$, for $0 < d_i < 1$ and that this item corresponds to an edge $e = \{u, v\}$. If both $d_u = d_v = 0$, then we can change the price of the edge item $e$ to $p'_e = M + 1$ without affecting the set of winning bids and increasing the total revenue. Therefore, assume that $d_u > 0$ and that $d_v 
less d_u$.

By Lemma 3, we know that $d_u + d_v < 1$ and thus $d_v < 1/2$. Consider the prices $p'_ı$ defined by $p'_ı = M + 1$, $p'_e = p'_v = M$, and $p'_v = p_v$ for all other items $ı$. The increase in revenue due to this price change for the two-item bundles corresponding to the edge $e = \{u, v\}$ is $1 - (d_u + d_v) - (d_u + d_v) - 2(d_u + d_v) - (3(d_u + d_v))$, whereas the total decrease in revenue for all other two-item bundles is bounded by $3(|E| - 1) (d_u + d_v)$. The increase in revenue caused by the single item bundles $e$ and $u$ is $(1 - d_u)(M + 1) - d_u (M + 1)$, whereas the increase in revenue for the single item bundles on $v$ is $M - (M + 1) d_v > 0$ if $d_v > 0$ and 0 otherwise. Hence, the total increase in revenue due to this price change is $1 - (d_u + d_v) - (d_u + d_v) - 2(d_u + d_v) - (3(d_u + d_v)) + (1 - d_u)(M + 1) + M - d_u (M + 1) + M - (3(d_u + d_v)) + (1 - d_u + d_v)(M + 1) + M - (3(d_u + d_v)) > 0$, as $M > 3 |E| - 1$. Hence, any solution in which an edge item $e$ has a fractional price cannot be optimal.

On the other hand, suppose that all edge items $e \in E$ have price $p_v \in \{M, M + 1\}$ and there exists a vertex item $u$ with $M < p_u < M + 1$. Then by Lemma 3 we know that there exists an item $u$ such that all its neighbors $v$ with $(u, v) \in E$ have price $p_v \leq M + 1/2$. Moreover, we may assume that there exists a neighbor $v$ such that $p_v > M$ as otherwise we may set $p_u = M + 1$ without affecting the set of winning bids, but increasing the total revenue. Let $d = \max (d_v : (u, v) \in E)$, then $0 < d < 1/2$. Consider the prices $p'_ı$ defined by $p'_ı = M + 1$, $p'_e = p'_v = M$ for $v$ such that $(u, v) \in E$ and $p'_v = p_v$ for all other items $ı$. Then the revenue for each two-item bundle is diminished by at most $2d$ and as there are in total $4|E|$ two-item bundles the total decrease in revenue due to this price change is bounded by $4|E| (2d^2) \leq 4|E|$. As there was at least one neighbor of $u$ that had a price $p_u > M$, the change in price leads to an increase of revenue of at least $(1 - d_u)(M + 1) - d_u (M + 1) + (1 - d_u - d_v)(M + 1) + M \geq M$, as by Lemma 3 $d_u + d_v < 1$. Hence the total increase in revenue due to this price change is at least $4|E| > 0$, as $M < 4|E|$. Hence, no solution with at least one fractional price can be optimal. □

**Theorem 5.** The revenue maximization problem is strongly NP-hard, even if the prices need to satisfy the monotonicity constraint.

**Proof.** Consider instance $I$. We claim that there exists an independent set of size $s$ in $G$ if and only if there is a solution to the revenue maximization problem with revenue at least $f(s) = (M^2 + 2M)|V| + (M^2 + 10M + 3)|E| + s$.

Given an independent set $V' \subseteq V$ with $|V'| = s$. Let $E_0 = \{e = \{u, v\} \in E : u, v \notin V'\}$. Let $p_v = M + 1$ if $v \in V'$ and $M$ otherwise. Let $p_v = M + 1$ if $e \in E_0$ and $M$ otherwise. The revenue of this solution is $\pi = (M + 1)(M|V| + |V'| + M|E| + E_0) + M|V| |V'| + 4|E| \in E_o + 8|E| + 3|E| \in E_o = (M^2 + 2M)|V| + (M^2 + 10M + 3)|E| + |V'| = f(s)$. For the converse, assume there is an optimal solution of the pricing problem with revenue at least $f(s)$. We know by Lemmas 2 and 4, that the price for any item is either $M$ or $M + 1$. Let $V' = \{v \in V : p_v = M + 1\}$, then $V'$ is an independent set in $G$, as by Lemma 3 we know that for any $e = \{u, v\} \in E$ with $u \in V'$, $p_u \leq M$. Define $E_0 = \{e = \{u, v\} \in E : p_u = M + 1\}$. Then for any $e = \{u, v\} \in E_0$, we have that $p_v = p_u = M$ by Lemma 3. As the revenue for the four two-item bundles corresponding to an edge $e \in E_0$ is bounded by $2(M + 1) + 2M = 8M + 3$ the total revenue is bounded from above by $|V'| (M + 1)^2 + |V'| (M^2 + 2M) + |E| (M + 1)^2 + |E| (M^2 + 2M) + |E| (2M + 1) + 2M + (8M + 3) = (M^2 + 2M)|V| + |V'| (M^2 + 10M + 3)|E|$. As the total revenue is at least $f(s)$, we conclude that $|V'| \geq s$, which finishes the proof. □

### 2.2. Approximation scheme

Assume, without loss of generality, that $p_1 \leq p_2 \leq \cdots \leq p_m$, then by the monotonicity constraint, we know $2p_1 \geq p_1 + p_2 \geq p_m$. Similarly, $3p_1 \geq p_1 + p_2 + p_3 \geq p_{m-1} + p_m \geq 2p_{m-1}$, etc.

**Lemma 6.** Suppose $p_1 \leq p_2 \leq \cdots \leq p_m$. Any pricing of the items satisfying the monotonicity constraint also satisfies

$$\frac{k}{p_k} \geq (k - 1)p_{m-2} - k, \quad k = 2, \ldots, \left\lfloor \frac{m}{2} \right\rfloor.$$

The idea for the PTAS is now the following. First, we restrict the prices to powers of $(1 + \delta)$ for some $\delta > 0$. Then, Lemma 6 says that except for a constant number of the cheapest and most expensive items, all items have prices in roughly the same range. Therefore we can price all except a constant number of items uniformly with the same price, without loosing too much in terms of the total revenue. We therefore enumerate over all possible uniform prices for the bulk of the items, and over all possible combinations of prices for the remaining (constant number of) items.

**Theorem 7.** The pricing problem with monotonicity constraint admits a PTAS. The time complexity is $O(nm^{8/3} / (\log B)^{8/3})$, where $B$ is the precision of the PTAS and $B = \max b_i$.

**Proof.** Given an instance of the pricing problem and an $\epsilon > 0$, let $\delta = \epsilon / 4$, and for convenience assume that $1 / \delta$ is integral. Let $p_1 \leq \cdots \leq p_m$ be the prices in an optimal solution, satisfying the monotonicity constraints. Define the subsets of items $S = \{i \in I : 1 \leq i \leq \frac{1}{\delta}\}$, $M = \{i \in I : \frac{1}{\delta} < i \leq m + 1 - \frac{1}{\delta}\}$ and $L = \{i \in I : i \geq m + 2 - \frac{1}{\delta}\}$. Note that $M = \emptyset$ if $\delta \leq 8 / (m + 1)$, in which case the number of items is in $O(1/\epsilon)$.

We round down the prices of all items in $S$ and $L$ to powers of $(1 + \delta)$. Moreover, we price all items in $A$ uniformly at price $p_1 + 1 / \delta$, rounded down to a power of $(1 + \delta)$. Let us call the new prices $p'$, and let us call $p_M$ the price of items in $M$. First observe that the order of prices does not change. We next argue that we do not lose too much by this rounding. Clearly, since we only round down, the set of winning bids can only increase. Moreover, we loose at most a factor $(1 + \delta)$ on items in $S$ and $L$. Finally, consider the items in $M$. By (2), we have

$$\left(1 + \frac{1}{\delta}\right)p_1 + 1 / \delta \geq \frac{1}{2} p_{m + 1 - 1 / \delta}.$$

In other words, the price for the most expensive item in $M$ differs from the cheapest item in $M$ by a factor at most $(1 + \delta)^2$. Hence, on items in $M$ we loose a factor at most $(1 + \delta)^2$. Now we have a structured solution, but it may violate the monotonicity constraint. In case of a violation, we divide the highest prices by a factor of $(1 + \delta)$, so that the new price vector satisfies the monotonicity constraint. Note that in this case, we loose at most a factor of $(1 + \delta)^3$ compared to the optimal pricing vector. The question remains which prices need to be divided by $(1 + \delta)$. To this end, we define $s = \max \{i : p_i < p_M\}$ as
the number of small prices and $\ell = \max\{0, i : p_{m+1-i} > p_{m}\}$ as the number of high prices. If $s > \ell$, then the new prices are defined by $p_i' = p_i$ for $1 \leq i \leq s$ and for $s + 1 \leq i \leq m$ by $p_i' = p_i/(1 + \delta)$. Otherwise, if $s \leq \ell$, then the new prices are defined by $p_i' = p_i$ for $1 \leq i \leq m - \ell$ and for $m - \ell + 1 \leq i \leq m$ by $p_i' = p_i/(1 + \delta)$. Let $k = \max(s - 1, \ell)$. Then, if there is a violation of the monotonicity constraint in the new pricing, sets $\{1, \ldots, k + 1\}$ and $\{m + 1 - k, \ldots, m\}$ are the two most violating sets, as $p_{k+1} = p_{m-k} = p_m$. Therefore, we only need to verify that for these two sets the monotonicity constraint holds true. First notice that by the way of defining $k$ and the prices $p_i'$, we have that $p_i' = p_i$ for $1 \leq i \leq k + 1$ and $p_i' = p_i/(1 + \delta)$, for $m + 1 - k \leq i \leq m$. Hence, we have that

$$
\sum_{i=1}^{k+1} p_i' = \sum_{i=1}^{k+1} p_i \geq \sum_{i=m+1-k}^{m} p_i \geq \sum_{i=m+1-k}^{m} p_i/(1 + \delta)
$$

where the second inequality is due to the monotonicity of the original set of prices $p$. From this it follows that the structured solution $p'$ satisfies the monotonicity constraint.

The PTAS now consists of enumerating all possible structured solutions, which is sufficient to obtain a feasible solution that differs from the optimal solution by a factor at most $(1 + \delta)^3 < (1 + \epsilon)^3$ (see [12]). There are $\binom{m}{\ell-1+2/3}$ possible choices for $S \cup L$. Since all prices are powers of $(1 + \delta)$, there are $\log B$ possible prices. Given that all items in $M$ have the same price, there are at most $(\log B)^2/8$ structured solutions for each choice of $S \cup L$. Computation of the revenue for any such solution takes $O(nm)$ time. This together with $\delta = \epsilon/4$ yields the claimed time complexity, where the constant hidden in the $O$-notation depends on $\epsilon$. □

3. Special case: The highway problem

A particularly intriguing special case of the general pricing problem considered so far is the “highway problem” as introduced by Guruswami et al. [12]. Here, the items are edges of a simple path, and the bundles corresponding to bids requested by bidders are subpaths. The problem is known to be weakly NP-hard [5], and a $\log(m)$-approximation exists [2]. Recently, also a quasi-PTAS [7] has been derived. The existence of a constant-factor approximation algorithm, however, is still open.

In this setting, it is most natural to assume that the monotonicity constraint holds for any two subpaths only, but not necessarily for arbitrary subsets of items. Hence, the weak NP-hardness result and the PTAS for the revenue maximization problem with monotonicity constraint, established in the previous section, do not carry over to the highway problem with price monotonicity on subpaths. Also notice that the weak NP-hardness for the general highway problem from [5] does not automatically yield weak NP-hardness for the problem with price monotonicity as optimal prices in that completeness proof are not monotone in the length of the subpaths. We next derive weak NP-hardness for the problem with monotonicity constraint, and we present a simple $O(\log B)$-approximation algorithm, where $B = \max_{a \in A} b_j$.

3.1. Complexity

We prove NP-hardness of the highway problem with monotonicity constraint by a reduction from the problem EquaCalNDaLTYSubsetSum: Given a set of positive integers $a_1, a_2, \ldots, a_{2L}$ and nonnegative integer $A$, does there exist a set $S \subseteq \{1, \ldots, 2L\}$ such that $\sum_{a \in S} a = A$ and $|S| = L$? This problem is readily shown NP-complete by a simple reduction from SubsetSum (see [8]).

Consider an instance of EqualCardinalitySubsetSum with $0 < a_1 \leq a_2 \leq \cdots \leq a_L$ and assume that $a_2 > \sum_{i=1}^{2L-1} a_i$ and $0 < A < a_L$. These assumptions do not harm the NP-completeness of the problem, as we can always add two large items $a_{2L+1} = 2\sum_{i=1}^{2L} a_i$ and $a_{2L+2} = 2\sum_{i=1}^{2L} a_i + 1$ and set $A = a_{2L+1}$. Then a subset $S$ in the instance of EqualCardinalitySubsetSum exists if and only if $\sum_{a \in S} a + a_{2L+1} = A$. We now create an instance $\mathcal{H}$ of the highway problem as follows. For every $\ell = 1, \ldots, 2L$, we define $a_i = a_i + a_{2L+1}$, and we introduce a gadget. Every gadget $\ell$ consists of four items $i_1, i_2, i_3, i_4$. Furthermore, there are $4 + 8L$ bids in every gadget $\ell$, one bid is on bundle $\{i_1, i_2\}$ with valuation $2M - \frac{1}{3}a_i'$, one bid is on bundle $\{i_3, i_4\}$ with valuation $a_i'$. 8L bids are on bundle $\{i_1, i_2, i_3\}$ with valuation $2M + \left(\frac{1}{2} + \frac{2}{3}a_i\right)$, one bid is on bundle $\{i_1, i_2, i_3\}$ with valuation $2M - \frac{2}{3}a_i'$, and one bid is on bundle $\{i_1, i_4\}$ with valuation $M$, where $M$ is a sufficiently large integer. For gadget 2L, there is one additional bid on bundle $\{i_1, i_2, i_3, i_4\}$ with valuation 5M. Finally, there is one big bid on all items with valuation $10M + \frac{4}{3}a_i + \frac{2}{3}A$. Thus, the instance of the highway pricing problem has 2L(4 + 8L) + 2 bids and 8L items, see Fig. 1.

We claim that there exists a set $S \subseteq \{1, \ldots, 2L\}$ such that $\sum_{a \in S} a = A$ and $|S| = L$ if and only if there is a feasible solution to the highway problem, fulfilling the monotonicity constraint, and with a total revenue $r \geq (20 + 32L)M + 5M + 4L\sum_{a \in S} a + (8L^2 + \frac{3}{2})a_{2L} + \frac{3}{2}A$, therefore proving the following theorem.

Theorem 8. The highway problem with monotonicity constraint is NP-hard.

Proof. First, given a set $S \subseteq \{1, \ldots, 2L\}$ such that $\sum_{a \in S} a = A$ and $|S| = L$. Let the price vector of gadget $\ell$ be $p_i' = (2M - \frac{1}{3}a_i, 2M - \frac{2}{3}a_i, 2M - \frac{1}{3}a_i', M)$ if $\ell \not\in S$, and $p_i' = (2M - \frac{1}{3}a_i, (1 + \frac{1}{2})a_i, 2M - \frac{1}{3}a_i', M)$ if $\ell \in S$. The revenue of every gadget $\ell$ is $(5 + 16L)M + 4L(a_i + a_{2L})$, independent of which price vector is used, and is given in revenue gadget $2L$ of 5M. Given the pricing strategy for all items, the big bid contributes $\sum_{a \in S} (5M + \frac{4}{3}(a_i + a_{2L})) + \sum_{a \not\in S} 5M = 10M + \frac{4}{3}a_{2L} + \frac{3}{2}A$. This yields a total revenue of $(20 + 32L)M + 5M + 4L\sum_{a \in S} a + (8L^2 + \frac{3}{2})a_{2L} + \frac{3}{2}A$.

For the converse, there is a feasible solution to the highway pricing problem with a total revenue of at least $r = (20 + 32L)M + 5M + 4L\sum_{a \in S} a + (8L^2 + \frac{3}{2})a_{2L} + \frac{3}{2}A$. It can be shown that in any solution to $\mathcal{H}$ with revenue at least $r^*$, the items in gadget $\ell$ must be priced according to either price vector $p_i'$ or $p_i''$ as defined above. Moreover, it can be shown that any pricing strategy that only uses price vectors $p_i'$ and $p_i''$ for each gadget satisfies the monotonicity constraint. Notice also that to obtain the maximum revenue on gadget 2L, we use pricing $p_i''$. We refer to [12] for details. Now, let set $S = \{\ell : p_i = p_i''\}$. Then, the payment of the big bid can be written as $\sum_{a \in S} (5M + \frac{4}{3}(a_i + a_{2L})) + \sum_{a \not\in S} 5M = 10M + \frac{4}{3}S_{a_{2L}} + \frac{3}{2}A$. In any solution to $\mathcal{H}$ with revenue at least $r^*$, one can verify that this payment must be equal to the valuation of the big bid, $10M + \frac{4}{3}S_{a_{2L}} + \frac{3}{2}A$; again see [12] for details. That is, $\sum_{a \in S} a = (L - |S|)a_{2L} + A$. We claim that $|S| = L$. To prove this, suppose it is not true. First, assume that $|S| < L$, then $\sum_{a \in S} a = (L - |S|)a_{2L} + A > \sum_{a \not\in S} a + A$, which is not possible as $2L \not\in S$, so $|S| > L$. Now, assume $|S| > L$, then $\sum_{a \in S} a = (L - |S|)a_{2L} + A < -a_{2L} + A < 0$, as $A < a_{2L}$. This is also not possible as all integers $a_i$ are nonnegative. Therefore, we can conclude that $|S| = L$ and consequently, $A = \sum_{a \in S} a$. □
3.2. Approximation algorithm

Notice that we cannot apply the PTAS from Theorem 7 to
the highway problem, as this crucially requires the monotonicity
constraint for arbitrary subsets of items. Nevertheless, we derive
an $O(\log B)$-approximation algorithm for the highway pricing
problem with monotonicity constraint, where $B = \max_j b_j$. To this
end, we present approximation guarantees for two special cases
first.

**Lemma 9.** The highway pricing problem with monotonicity con-
straint in which all bundles have size at least two is approximable
within a factor of 3 by optimal uniform pricing.

**Proof.** Consider an optimal solution with revenue $\text{orr}$ and let $P^*_{\text{max}}$
be the highest item price in this solution. We claim that pricing
all items at $P^*_{\text{max}}/3$, yields a revenue of at least $\text{orr}/3$. Clearly, an
optimal uniform pricing is at least as good as the uniform $P^*_{\text{max}}/3$
pricing.

First, we show that any winning bid $j \in W$ in the optimal pricing
remains a winning bid for the uniform pricing at level $P^*_{\text{max}}/3$. Let
$|b_j| = \ell$. Then the valuation for bid $j$ is at least $b_j \geq \lfloor \ell/2 \rfloor P^*_{\text{max}}$, as by
the monotonicity constraint the total price of any two consecutive
items in an optimal solution is at least $P^*_{\text{max}}$, and the bidder who
places bid $j$ can afford the corresponding bundle $l_j$. In the uniform
$P^*_{\text{max}}/3$ pricing, the total bundle price is $\ell P^*_{\text{max}}/3$, which is at most
$\lfloor \ell/2 \rfloor P^*_{\text{max}}$ for $\ell \geq 2$. In an optimal pricing, bundle $l_j$ corresponding
to bid $j$ is priced at most at $\ell P^*_{\text{max}}$, whereas in our uniform pricing,
we get $\ell P^*_{\text{max}}/3$. Hence, pricing all items $P^*_{\text{max}}/3$ yields a revenue of
at least $\text{orr}/3$. □

The above lemma shows that whenever all bundles contain
at least two items, we have a constant approximation. Now, we
consider only instances in which bundles consist of exactly one
item. Moreover, we restrict ourselves to instances in which $b_j/b_k \leq
2$, for any two bids $j$ and $k$.

**Lemma 10.** The highway pricing problem with monotonicity con-
straint, restricted to instances in which each bid is on a bundle with
exactly one item and $b_j/b_k \leq 2$ for any two bids $j, k$, admits a linear
time 2-approximation algorithm

**Proof.** We price each item uniformly at $p = \min_j b_j$. Since for any
two bids $j, k$, it holds that $b_j/b_k \leq 2$, and every bid is on a bundle
with one item, we loose at most a factor of 2 for every bid. Hence,
pricing all items at $p$ yields a revenue of at least $\text{orr}/2$. □

**Theorem 11.** The best uniform pricing yields a solution with revenue
at least $\text{orr}/(3+2\log B)$ for the highway problem with monotonicity
constraint, where $B = \max_j b_j$. Moreover, the time needed to find this
solution is $O(n^2 m)$.

**Proof.** Consider an optimum solution satisfying the monotonicity
constraint, and let $\text{orr}_j$ denote the revenue of bidders whose bids
are on bundles of size at least two and let $\text{orr}_j$ denote the revenue
of bidders whose bids are on bundles of size one with valuation
$2^{-r} \leq b_j < 2^{r} (r = 1, \ldots, \lfloor \log_2 B \rfloor + 1$) in this solution. Then
$\text{orr} = \text{orr}_1 + \text{orr}_2 + \cdots$. Moreover, let $\text{app}_j$ denote the revenue
obtained by the best uniform pricing and $\text{app}_j$ denote the revenue
obtained by the best uniform pricing strategy for the bidders with
bids in $J_j = \{ j \in J : |b_j| = 1 \text{ and } 2^{-r} \leq b_j < 2^{r} \}$. By Lemma 10, we
have that $\text{app}_j \geq \text{orr}_j/2$ and thus $\text{app} = \sum \text{app}_j$. Moreover, from
Lemma 9, it follows that $\text{app} \geq \text{orr}/3$. Hence,
the solution found yields a revenue of $\text{orr}/(3+2\log B)$.

To see the claim on the time complexity, note that to find
an optimal uniform pricing, we need to consider at most $O(n)$
different prices, namely $b_j/|b_j|$ for all $j \in J$. For each price, we need
to compute the set of winning bids and the revenue obtained on
this price, which can be done in $O(nm)$ time. So, the best uniform
price can be computed in $O(n^2 m)$ time. □

4. Conclusion

This paper studies omniscient pricing problems, reflected by
the fact that we assume bidders’ valuations $b_j$ to be known.
Even more challenging are problems where valuations are private
information, and incentive-compatible (truthful) mechanisms are
sought. To that end, we refer to Goldberg et al. [9] or Balcan et al. [3],
and note that Balcan et al. [4], extending results by Guruswami et al. [13],
present a truthful mechanism with performance guarantee $O(\log n + \log m)$
for single item pricing in combinatorial auctions with unlimited supply, hence particularly
for the model considered in this paper. Again, the underlying
pricing mechanism is very simple; it consists of randomly
choosing a uniform price for all items. Extending more involved
approximation algorithms to truthful mechanisms remains an
interesting problem for future research.

Acknowledgements

Joyce van Loon acknowledges support by METEOR, the Maas-
tricht Research School of Economics of Technology and Organiza-
tions. We also thank the anonymous referee for careful reading
and helpful remarks.

References

pricing, in: J. Diaz, J. Karhumäki, A. Lepistö, D. Sannella (Eds.), Automata,
Languages and Programming — ICALP 2004, in: Lecture Notes in Computer


